Brauer group of Lubin–Tate spectra

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I Motivation and context

Say, we want to classify skew-fields D

It is an (division) algebra over its center K, which is a field

Need to classify division algebras $/K$ with center K (central division algebras)

up to isom for fields K. (assume $\dim_K D < \infty$)

Problem: This is an unstructured set.

Azumaya algebras and Brauer groups

Def: Let $\mathcal{C}$ be a sym mon presentable ∞-cat (e.g. $\mathrm{Mod}_K^{\otimes}$).

- Two algebras A, B in $\mathcal{C}$ are Morita equivalent if Mod_A and Mod_B are equivalent as $\mathcal{C}$-lin categories (i.e. as $\mathcal{C}$-modules in Pr^L)

- An algebra A in $\mathcal{C}$ is Azumaya if there exists B s.t. $A \otimes B \underset{\text{Morita}}{\simeq} 1\!\!1_{\mathcal{C}}$.

<u>Def</u>: $Br(\mathcal{C})$ = Azumaya algebras in $\mathcal{C}$ / up to Morita equivalence.

abelian group

<u>Prop</u>: A Azumaya iff

- dualizable
- faithful (ie. $\otimes A$ is conservative)
- $A \otimes A^{op} \xrightarrow{\simeq} \underline{End}_{\mathcal{C}}(A)$

  "$x \otimes y$" $\longmapsto$ $(a \mapsto xay)$

<u>Prop</u>: Let $\mathcal{C} = Mod_k^{\otimes}$

a) Central division algebras are Azumaya

b) If $D \underset{\text{Morita}}{\sim} D'$, then $D \simeq D'$

c) If $A \in Alg(Mod_k^{\otimes})$ is Azumaya, then $A \simeq Mat_{n \times n}(D)$ for some n

$\underset{\text{Morita}}{\sim} D$

In particular: $\mathrm{Br}(\mathrm{Mod}_k^{\omega}) \cong$ centr. div algebras / isom

<u>Proof of 6</u>: Suppose $D \underset{\mathrm{Mor}}{\sim} D'$

$$\rightsquigarrow \quad F : \mathrm{Mod}_D \xrightarrow{\simeq} \mathrm{Mod}_{D'}$$

D is indecomposable in Mod_D $\Rightarrow$ $F(D) \in \mathrm{Mod}_{D'}$ is indecomposable

$\Rightarrow F(D) \cong D'$ in $\mathrm{Mod}_{D'}$

$$D' = \mathrm{End}_{\mathrm{Mod}_{D'}}(\overset{F(D)}{\widetilde{D'}}) = \mathrm{End}_{\mathrm{Mod}_D}(D) \cong D. \qquad \blacksquare$$

Set $\mathrm{Br}(k) = \mathrm{Br}(\mathrm{Mod}_k^{\omega})$ for k a commutative ring.

<u>Example</u>: $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2 = \{\mathbb{R}, \mathbb{H}\}$

$\mathrm{Br}(\mathbb{C}) = 0$ $\qquad$ $\mathrm{Br}(\mathbb{Z}) = 0$ $\qquad$ $\mathrm{Br}(k) = H^2(\mathrm{Gal}(\bar{k}/k); \bar{k}^\times)$

$\mathrm{Br}(\mathbb{F}_q) = 0$ $\qquad$ $\mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$

<u>Antieau-Gepner</u>: If R connective ring spectrum, then

$\mathrm{Br}(\mathrm{Mod}_R) \cong \mathrm{Br}(\pi_0 R)$ $\quad$ (if $\pi_0 R$ is reasonable, i.e. at least normal)

$\mathrm{Br}(\mathbb{S}) = \mathrm{Br}(\mathbb{Z}) = 0$

If R non-connective, $Br(Mod_R)$ is tough to study,

e.g. $Br(Mod_{KU}) = 0$?? seems unsolved.

$Br(\mathbb{Z}) = 0$

What about $Br(Mod_{KU_p^\wedge})$?

$K(1) = KU/p$ has an A_∞-KU-alg. structure. Can it be Azumaya? No, not faithful

Consider instead: $Br(Mod^{loc}_{KU_p^\wedge})$ — $K(1)$-loc $\Leftrightarrow$ p-complete

$K(1) \in Mod^{loc}_{KU_p^\wedge}$ is faithful and dualizable,

can be Azumaya

More generally, Hopkins-Lurie study

$Br(Mod^{loc}_{E_n}) =: Br(E_n)$ — $K(n)$-local

└ Lubin-Tate spectrum for a formal group law of height n over a field κ

They call an E_n-algebra A atomic if $\pi_0 A = \kappa$, $\pi_1 A = 0$

so essentially A is a form of Morava K-theory

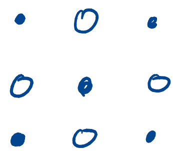
$\rightsquigarrow \pi_* A \cong \kappa[u^{\pm 1}]$

Remark: There are many atomic E_n-algebras, but they are all equivalent as E_n-modules.

Indeed: Let A, B be atomic E_n-algebras.

$$\mathrm{Ext}^{s,t}_{\pi_* E_n}(\pi_* A, \pi_* B) \Rightarrow \pi_{t-s}\mathrm{Hom}_{E_n}(A, B)$$

Because $\pi_* A, \pi_* B$ are conc. in even degrees, have checkerboard phenomenon



$\rightsquigarrow$ no differentials

$\Rightarrow$ we can lift $\mathrm{id}: \pi_* A \to \pi_* B$ to a map $A \xrightarrow{\simeq} B$.

It will be again the case that some atomic E_n-algebras are Azumaya, others are not.

Prop: If A, B are Morita-equivalent atomic E_n-Azumaya algebras, then $A \underset{E_n\text{-alg}}{\simeq} B$.

Proof is the same as for division algebras, using that A is the only indec object in $\mathrm{Mod}_A^{\mathrm{ad}\, A[1]}$

since $\mathrm{ho}(\mathrm{Mod}_A) \simeq \mathbb{Z}/2$-gr modules over π_0.

Summary of goals

1) Calculate $\mathrm{Br}(E_n)$

2) Identify the image of atomic Azumaya algs in there $\rightsquigarrow$ classification of atomic Azumaya algebras up to equivalence.

II Methods and results of Hopkins–Lurie

$\mathrm{Br}(E) = \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}})$ hard to study as ($E = E_n$) $\mathrm{Mod}_E^{\mathrm{loc}}$ doesn't have a helpful t-structure.

Can we replace $\mathrm{Mod}_E^{\mathrm{loc}}$ by a prestable ∞-category with the same Brauer group?

Yes.

IIa Synthetic E-modules

Let $\mathrm{Mod}_E^{\mathrm{mol}} \subseteq \mathrm{Mod}_E$ be the full subcat on finite sums of shifts of atomic E-modules.

(If K atomic: $\mathrm{Mod}_E^{\mathrm{mol}} \simeq$ full subcat on $K^m \oplus \Sigma K^n$ for $m, n \geq 0$.)

A <u>synthetic E-module</u> is a finite-product pres functor $(\mathrm{Mod}_E^{\mathrm{mol}})^{op} \longrightarrow \mathcal{S}$.

$\rightsquigarrow \mathrm{Syn}_E$, <u>prestable</u>

$\nu: \mathrm{Mod}_E^{\mathrm{loc}} \longrightarrow \mathrm{Syn}_E$ "Yoneda embedding"

fully faithful, conservative and symmetric monoidal.

<u>Key result</u>: $\mathrm{Br}(E) \cong \mathrm{Br}(\mathrm{Syn}_E)$

<u>Idea</u>: $\nu: \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ restricts to an equivalence $(\mathrm{Mod}_E^{\mathrm{loc}})^{\text{dual, (faithful)}} \longrightarrow (\mathrm{Syn}_E)^{\text{dual, (faithful)}}$

b The heart (p+2)

$\mathrm{Syn}_E^{\heartsuit}$ = fin. prod-pres functors $h\mathrm{Mod}_E^{\mathrm{mol},op} \longrightarrow \mathrm{Set}$

This inherits a sym mon str. $\boxtimes$ from Syn_E.

Prop: $\mathrm{Syn}_E^{\heartsuit} \xrightarrow{\simeq} \mathrm{grMod}_{\pi_* \mathrm{End}_E(K)}$ (atomic E-algebra)

$M \longmapsto \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbb{1}^{\heartsuit}, K^{\heartsuit} \boxtimes M)$

Sym mon equivalence. $\mathbb{1} = \mathrm{Sy}(E) \in \mathrm{Syn}_E$

What is $\pi_* \mathrm{End}_E(K)$?

<u>UCSS:</u> $\mathrm{Ext}^{s,t}_{\pi_* E}(\pi_* K, \pi_* K) \Rightarrow \pi_{t-s} \mathrm{End}_E(K)$

$\pi_0 K = k = \pi_0 E / (v_0, v_1, \dots, v_{n-1})$ (reg. seq.) $\quad$ ($n=1$: $\pi_0 E = \pi_0 \mathrm{KU}_p^{\wedge} = \mathbb{Z}_p$, $v_0 = p$)

Can calculate $\mathrm{Ext}^*_{\pi_0 E}(k, k)$ via Koszul complex

One uses a resolution:

$$0 \to \Lambda^n_{\pi_0 E}((\pi_0 E)^n) \to \Lambda^{n-1}_{\pi_0 E}((\pi_0 E)^n) \to$$

$$\to \cdots \to \Lambda^0_{\pi_0 E}((\pi_0 E)^n) \to \kappa \to 0$$

Homing into κ produces a complex

$$0 \to \Lambda^n_\kappa(\kappa^n) \to \Lambda^{n-1}_\kappa(\kappa^n) \to \cdots \to \Lambda^0_\kappa(\kappa^n)$$

with 0 differentials.

$$\Rightarrow \operatorname{Ext}^s_{\pi_0 E}(\kappa, \kappa) = \Lambda^s(\kappa^n)$$

solving ext issues $\rightsquigarrow$ $\pi_* \operatorname{End}_E(\kappa) = \Lambda^*(\kappa^n)[u^{\pm 1}]$

<u>Upshot:</u> $\operatorname{Syn}_E^\heartsuit \simeq \mathbb{Z}/2\text{-gr}\operatorname{Mod}_{\Lambda^*(\kappa^n)}$

κ^n ⤳ m/m^2 if $m \subseteq \pi_0 E$ max ideal

<u>Prop:</u> $\operatorname{Br}(\operatorname{Syn}_E^\heartsuit) \simeq \operatorname{BW}(\kappa) \times QF(m/m^2)$

$\operatorname{BW}(\kappa)$: $\operatorname{Br}(\mathbb{Z}/2\text{-graded } \kappa\text{-modules})$

$\operatorname{BW}(\mathbb{R}) = \mathbb{Z}/8$, $\operatorname{Br}(\mathbb{F}_p) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$ depending on p mod 4.

c The tower

Consider $\mathbb{1}^{\leq n} \in \mathrm{Syn}_E$, i.e. the n-truncation of $\mathbb{1} = \mathrm{Sy}(E)$.

$$\rightsquigarrow \mathrm{Br}(\mathbb{1}^{\leq n}) = \mathrm{Br}(\mathrm{Mod}_{\mathbb{1}^{\leq n}}(\mathrm{Syn}_E))$$

Prop: $\mathrm{Br}(\mathbb{1}^{\leq 0}) = \mathrm{Br}(\mathrm{Syn}_E^{\heartsuit})$

Thm: $\mathrm{Br}(\mathbb{1}^{\leq n}) \longrightarrow \mathrm{Br}(\mathbb{1}^{\leq n-1})$ surjective

with kernel $= \underbrace{\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3}}$

part of assoc. graded of $\pi_0 E$ with filtration by powers of $\mathfrak{m}$.

Prop: $\mathrm{Br}(\mathrm{Syn}_E) \simeq \varprojlim \mathrm{Br}(\mathbb{1}^{\leq n})$

Summary: $\mathrm{Br}(E) \simeq \mathrm{Br}(\mathrm{Syn}_E)$ has a filtration

with associated graded $\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3}$ for $n \geq 0$

(here I use: $QF(\mathfrak{m}/\mathfrak{m}^2) \equiv \mathfrak{m}^2/\mathfrak{m}^3$)

and $\mathrm{BW}(k)$ as the bottom piece.

E.g. for $E = Ku_p^1$, up to extension issues,

$$\mathrm{Br}(E) \text{ is } \left(\underset{\substack{\wr\wr \\ \mathbb{Z}_p}}{(p)} \subseteq \mathbb{Z}_p \right) \oplus \mathrm{BW}(\mathbb{F}_p)$$

In particular, $\mathrm{Br}(E)$ is uncountable for $n \geq 1$.

d. <u>Where are the atomic Azumaya algebras</u>

<u>Thm</u>: Let $x \in \mathrm{Br}(E) = \mathrm{Br}(\mathrm{Sp}_E)$

a) Whether x is represented by an atomic Azumaya alg, only depends on the image in $\mathrm{Br}(\mathrm{Sp}_E^\heartsuit)$

(Recall: $\mathrm{Br}(\mathrm{Sp}_E^\heartsuit) = \mathrm{BW}(k) \times \mathrm{QF}(\overbrace{\mathfrak{m}/\mathfrak{m}^2}^{V})$

Given a quadratic form q on V, we can construct its Clifford algebra $\mathrm{Cl}(q) = T(V) / \underset{x \in V}{x^2 = q(x)}$

If q is non-deg, this is in $\mathrm{BW}(k)$.
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b) An element in

$$\mathrm{Br}(\mathrm{Sym}_E^{\otimes}) = \mathrm{BW}(k) \times \mathrm{QF}\left((\mathfrak{m}/\mathfrak{m}^2)^{\vee}\right)$$

is represented by an atomic Azumaya alg iff it is of the form $(Cl(q), q)$, q non-deg.