

Adams spectral sequences

Thm. Let E be Morava theory and $p \gg n$. Then:

$$hSp_E \cong hD(Comod_{E,E}).$$

E -local spectra

derived category of differential comodules

Idea:

Resolve Sp_E using Goren-Hopkins theory.



For $p \gg n$:

- $Ext_{E,E}^s = 0$ for $s \geq n^2 + n \Rightarrow$

a range in which $hSp_E \cong hM_k$

- $Comod_{E,E}$ is $2p-2$ -sparse (i.e. E,E is in deg. div. by $2p-2$).

$\coprod_{k \in \mathbb{Z}/(p-2)} Comod_{E,E}^k$ a range in which

M_k are algebraic.

For $p \gg n$: there's a non-zero intersection

Q: Where does the G-H tower arise from?

Idea: Left $E_n = Sp_E \rightarrow Comod_{E,E}$ to a functor into a

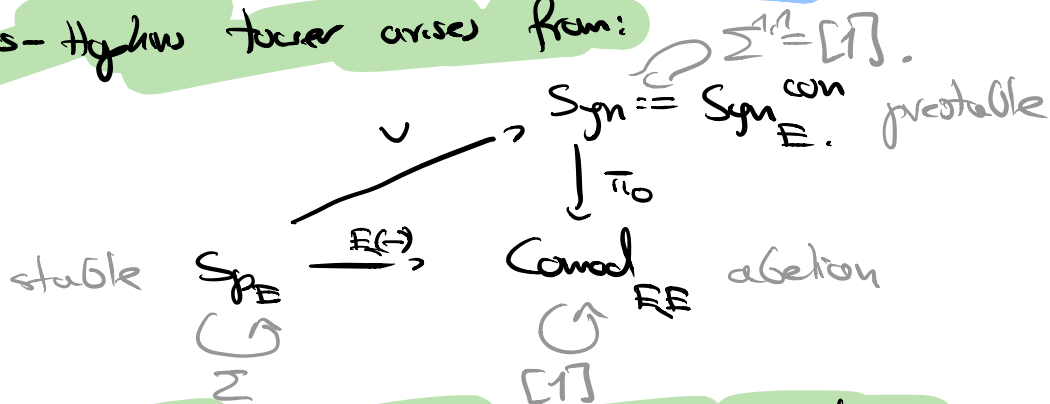
more structured ∞ -category.

Def. (Lurie) We say D is prestable if it is equivalent to a connective part of a t-structure on a stable ∞ -category.

META-THEOREM: Prestable ∞ -cats are the ∞ -categorical analogue of abelian categories.

- D -prestable $\Leftrightarrow D^{\text{op}} = \sum_{\mathbb{Z}} D$ - abelian.
- $\pi_k d = \sum_{\mathbb{Z}} \mathcal{O}^k d, k \geq 0$. subset of disc. objects
- A -abelian $\sim \text{DCA}^{\text{con}}$ prestable + universal prestable with $\mathcal{O} \geq A$.

Goerss-Hopkins tower arises from:



The functor ν has the following properties:

- $\pi_0 \circ \nu \cong E(-)$
- ν is left exact
- preserves infinite direct sums
- $\nu(\mathcal{O}X) \cong \nu(X)[1]$.

ν is a prestable enhancement to the homology theory $E_{\mathbb{Z}}(-)$.

Lemma: If $X \rightarrow Y \rightarrow Z$ is a cofibre seq. of spectra then $\nu X \rightarrow \nu Y \rightarrow \nu Z$ is cofibre iff $E_2 Y \rightarrow E_2 Z$ is surjective.

Proof: By left exactness, $\nu_X \rightarrow \nu_Y \rightarrow \nu_Z$ is fibre. This will be fibre iff. $\pi_0 \nu_Y \rightarrow \pi_0 \nu_Z$ is surjective.

Note: $\pi_h \nu_X \cong \pi_0 \Omega^h \nu_X \cong \pi_0 \nu \Omega^h X \cong \Gamma(\nu X) \Omega^h$.

The Goerss-Hopkins tower is of the form:

$$\begin{array}{ccccccc} \mathrm{Sym} & \rightarrow & \dots & \rightarrow & \mathcal{J}_2 \mathrm{Sym} & \rightarrow & \mathcal{J}_3 \mathrm{Sym} & \rightarrow & \mathcal{J}_\infty \mathrm{Sym} \\ \cup & & & & \cup & & \cup & & \cup \\ \mathrm{SpE} & \rightarrow & M_\infty & \rightarrow & M_2 & \rightarrow & M_1 & \rightarrow & M_0 \end{array}$$

The needed properties of the G-H tower follow from properties of Sym .

jav. Paul Van Koughnett
"Abstract G-H theory!"

Thm: (Frankel's algebraicity conjecture, Patchkoria-P.).

Let $H: C \rightarrow A$ be a homology theory admitting Adams stable abelian spectral sequences such that:

- A is of finite homological dimension d .
- we have a weight decomposition $A \cong \prod_{\ell \in \mathbb{Z}/N\mathbb{Z}} A^\ell$
- $N > d$

Then $hC \cong hDGA$.

Ex: $h\mathrm{SpE} \cong hDCE$ when $2p-2 > n^2+n$

Ex: R -ring spectrum, $\pi_* R$ concentrated in deg. div. of finite dim. $d < N$.

$h\mathrm{Mod}(R) \cong hDCR$.

Ex: ∞ -categories of sheaves, diagrams, filtered objects.

The key step in construction of Syn^* :

$$\text{Comod}_{\mathbb{F}E} \simeq \text{Sh}_2(S_p^{\text{fp}}, AB)$$

algebraic topological

Q: Which abelian categories can be described in this way?

Def: We say a homology theory $H: C \rightarrow A$ is adapted (admits the ASS) if:

- 1) A has enough injectives
- 2) for every injective $I \in A$, $\exists I_C \in C$ st. $[-, I_C] \simeq \text{Hom}_A(H(-), I)$. } usually automatic.

3) the canonical map $H(I_C) \rightarrow I$ is an iso.

We get an ASS. ↑ very much not automatic.

$$X := X_0 \subset I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} \dots$$

\uparrow
 H -mono into an H -injective. \downarrow \downarrow
 I_0/X_0 I_1/X_1

H -Adams resolution.

of signature

$$\text{Ext}_A^s(H(Y), H(X)) \Rightarrow \{Y, X\}_{+s}$$

Ex: $E_s: S_p \rightarrow \text{Comod}_{\mathbb{F}E}$ where E is Adams-type

Ex: $T_k: \text{Mod}(R) \rightarrow \text{Mod}(R_k)$

Ex: (The universal homology thry).

} finitely presented.

C -stable, $A(C) := \text{tan}_{\Sigma}(C; \text{Freyd envelope})$, $y: C \rightarrow A(C)$, $y(X)(Y) = [Y, X]$.

$C \xrightarrow{y} A(C)$

$\downarrow \checkmark \exists!$ (continuous, exact functor of abelian cats.)
 A .

Thm: Let $H: C \rightarrow A$ be a homology theory such that A has enough injectives. Then, TFAE:

- 1) H is adapted
- 2) there exists a localizing subcat. $\mathcal{L} \subseteq A(C)$ st. $A \simeq A(C)/\mathcal{L}$.
- 3) $A(C) \rightarrow A$ admits a fully faithful right adjoint.

Cor: Adapted homology theories out of C form a poset.

(dual of the poset of localizing subcats. of $A(C)$).
 lattice of ASSs.

Q: What does this convergence look like?

Def: We say $M \subseteq \text{Fun}(A^{\text{op}}, C)$ is a mono class:

- 1) eqv. are in M
- 2) $f, g \in M \Rightarrow f \circ g \in M$, $f \circ g \in M \Rightarrow g \in M$
- 3) M is stable under pushouts
- 4) M is stable under cusp.

We say $I \in C$ is M -injective if it has RLP

wt = to maps in \mathcal{M} .

Ex: $H = C \rightarrow A$, $\mathcal{M} := H$ -monic maps
 $\mathcal{M}(ij) :=$ objects of the form I_C .

Ex: R -ring spectrum
 $\mathcal{M} := (R \otimes -)$ -split injective maps
 $\mathcal{M}(ij) :=$ retracts of inclusion of a direct summand.
 R -modules.

Adams resolution: $X \rightarrow R \otimes X \rightarrow R \otimes R \otimes X \rightarrow \dots$

In general different from the E_1 -based ASS even when E is Adams-type.

Q: How to describe the E_2 -term in this example?

Thm: There's a 1-1 correspondence between mono classes on C st. C has enough inj. and localizing subsets of $A(C)$.

Cor: Every mono class arises uniquely as the class of H -monic maps for an adapted homology theory H .

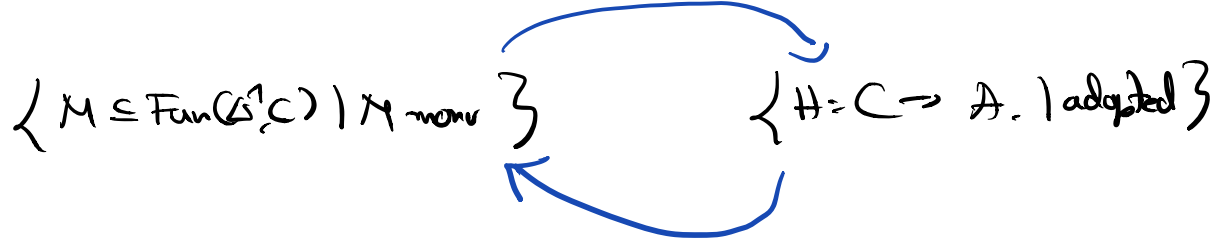
Every ASS has an E_2 -page an Ext group in appropriate abelian category.

Reht: In recent work, Balmer studies localizing ideals in $A(C)$ (homological residue fields).

Notation: We call fibres of \mathcal{M} -monic maps \mathcal{M} -zero fibres -1- \mathcal{M} -epic.

$$\dots \rightarrow \sum_{M=200U}^{-1} \dots \rightarrow X \xrightarrow{f \in M} Y \xrightarrow{M=200C} Z \xrightarrow{M=200} X \rightarrow \dots$$

$$A := \text{Sh}(C, M=200C) \text{ fp}$$



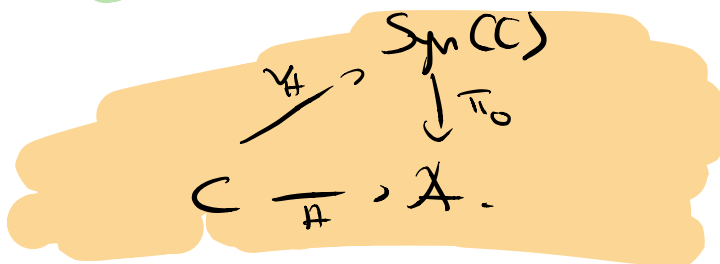
$M := H\text{-mono maps}$

Fact: $\text{Sh}_{\text{op}}(C) \subseteq P_{\text{op}}(C)$.
 this has an exact left adjoint when restricted to the subcats. of n -truncated objects for any fixed n .

Def: $\text{Syn}(C) := \varprojlim_{\leftarrow} \text{Sh}_{\text{op}}(C)$

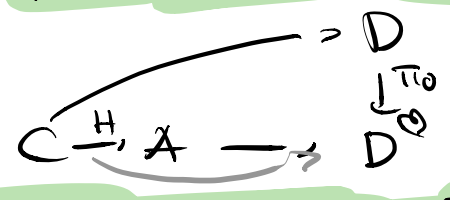
Ex: $\mathbb{F}_x = \text{Sp}_{\mathbb{F}} \rightarrow \text{Comod}_{\mathbb{F}} \Rightarrow \text{Syn}(\text{Sp}_{\mathbb{F}}) := (\text{Syn}_{\mathbb{F}})^{\wedge}$

Ex: $\mathbb{T}_x = \text{Mod}(R) \rightarrow \text{Mod}(R_x), \text{Syn}(\text{Mod}(R)) := \prod_{\Sigma} (\text{Mod}(R))^{\text{ff}}$



where v_H is a prestack enhancement to H .

Thm: The ∞ -cat. $\text{Syn}(C)$ is the universal prestack enhancement of H , i.e. for any diagram



there's a unique $\text{Syn}(C) \rightarrow D$ st. everything commutes.

universal property $D(A)^{\text{con}}$
 universal prealgebra D
 with a functor $A \rightarrow D$

$\text{Syn}(C)$
 universal prealgebra D
 with a functor $A \rightarrow D$
 + prealgebra enrichment
 of the composite Hof.

Frankel's algebraicity



$\dim(A) < \infty \Rightarrow$ there's a top. range in which $hC \cong hM_k$.

Q: How do we show M_k are algebraic for loc 1?

Key: Need an appropriate functor

$$\beta^1: \sum_{k \in \mathbb{N}} D(A^k) \rightarrow \sum_{k \in \mathbb{N}} \text{Syn}(C^k).$$

enough to define it on A .

$$h_{\mathbb{N}} C^{inj, \ell} \cong A^{inj, \ell} \Rightarrow$$

Bousfield splitting $A^{inj} \xrightarrow{\beta} h_{\mathbb{N}} C^{inj}$. We put

$$\beta^x(c) = (\sum_{i \in \mathbb{N}} \beta(c^i)).$$

For general $a \in A$, choose an inj. resolution $a \rightarrow c^\bullet$, put

$$\beta^x(a) = \text{Tot}(\beta^x(c^\bullet)).$$