Adams spectral sequences

Thm. Let E be Morava theory and $p \gg n$. Then:

$$h\mathrm{Sp}_E \simeq hD(\mathrm{Comod}_{E_*E}).$$

E-local spectra — derived ∞-cat of differential comodules.

Idea:

Resolve Sp_E using Goerss-Hopkins theory.

$$M_\infty \to \cdots\cdots \to \cdots \to M_2 \to M_1 \to M_0$$

$M_\infty \simeq \mathrm{Sp}_E$ $\qquad$ $M_0 \simeq \mathrm{Comod}_{E_*E}$

we have obstructions in Ext_{E_*E} to lifting obj / morphisms (htpies ...

For $p \gg n$:

- $\mathrm{Ext}^s_{E_*E} = 0$ for $s > n^2+n \implies$ a range in which $h\mathrm{Sp}_E \simeq hM_k$
- Comod_{E_*E} is $2p-2$-sparse (i.e. E_*E is in deg. div. by $2p-2$).
  $\mathrm{Comod}_{E_*E} \simeq \prod_{\varphi \in \mathbb{Z}/2p-2} \mathrm{Comod}^{\varphi}_{E_*E}$ $\implies$ a range in which M_k are algebraic.

For $p \gg n$: there's a non-zero intersection.

Q: where does the G-H tower arise from?

Idea: Lift $E_*: \mathrm{Sp}_E \to \mathrm{Comod}_{E_*E}$ to a functor into a more structured ∞-category.

Def. (Lurie) We say D is <u>prestable</u> if it's equivalent to a connective part of a t-structure on a stable ∞-category.

<u>META-THEOREM</u>: Prestable ∞-cats are the ∞-categorical analogue of abelian categories.

- D-prestable $\rightsquigarrow$ $D^{\heartsuit} := \tau_{\leq 0} D$ – abelian. (subcat of disc. objects)
- $\pi_n d = \tau_{\leq 0} \Omega^n d$, $n \geq 0$.
- A-abelian $\rightsquigarrow$ $D(A)^{cn}$ prestable + universal prestable with $\heartsuit \simeq A$.

Goerss–Hopkins tower arises from:

$$\Sigma^{1,1} = [1].$$

$$\begin{tikzcd} & \mathrm{Syn} := \mathrm{Syn}_E^{cn} \arrow[d, "\pi_0"] & \text{prestable} \\ \text{stable} \quad \mathrm{Sp}_E \arrow[ur, "\nu"] \arrow[r, "E_*(-)"] \arrow[loop below, "\Sigma"] & \mathrm{Comod}_{E_*E} \arrow[loop below, "{[1]}"] & \text{abelian} \end{tikzcd}$$

The functor ν has the following properties:

- $\pi_0 \circ \nu \simeq E_*(-)$
- ν is left exact
- preserves infinite direct sums
- $\nu(\Sigma X) \simeq \nu(X)[1]$.

$\Bigg\}$ ν is a prestable enhancement to the homology theory $E_*(-)$.

<u>Lemma</u>: If $X \to Y \to Z$ is a cofibre seq. of spectra then $\nu X \to \nu Y \to \nu Z$ is cofibre iff $E_* Y \to E_* Z$ is surjective.

Proof: By left exactness, $vX \to vY \to vZ$ is fibre. This will be cofibre iff. $\pi_0 vY \to \pi_0 vZ$ is surjective.

Note: $\pi_h vX \simeq \pi_0 \Omega^h vX \simeq \pi_0 v \Omega^h X \simeq (\pi_0 vX)[-h]$.

The Goerss-Hopkins tower is of the form:

$$\begin{tikzcd}
 & \mathrm{Syn} \arrow[r] & \cdots \arrow[r] & \tau_{\leq 2}\mathrm{Syn} \arrow[r] & \tau_{\leq 1}\mathrm{Syn} \arrow[r] & \tau_{\leq 0}\mathrm{Syn} \\
\mathrm{Sp}_E \arrow[r] & M_\infty \arrow[u, phantom, "\cup"] \arrow[r] & \cdots \arrow[r] & M_2 \arrow[u, phantom, "\cup"] \arrow[r] & M_1 \arrow[u, phantom, "\cup"] \arrow[r] & M_0 \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

The needed properties of the G-H tower follow from properties of Syn.

j.w. Paul VanKoughnett "Abstract G-H theory"

Thm: (Franke's algebraicity conjecture, Patchkoria-P.).

Let $H: C \to A$ be a homology theory admitting Adams spectral sequences such that:
stable abelian

- A is of finite homological dimension d.
- we have a weight decomposition $A \simeq \prod_{\ell \in \mathbb{Z}/N} A^\ell$
- $N > d$

Then $hC \simeq hD(A)$.

Ex: $h\mathrm{Sp}_E \simeq hD(E_*E)$ when $2p-2 > n^2+n$

Ex: R-ring spectrum, $\pi_* R$ concentrated in deg. div. by N, of finite dim. $d < N$.
$h\,\mathrm{Mod}(R) \simeq hD(R_*)$.

Ex: ∞-categories of sheaves, diagrams, filtered objects.

The key step in construction of Syn:

$$\underbrace{\mathrm{Comod}_{E_*E}}_{\text{algebraic}} \simeq \underbrace{\mathrm{Sh}_\Sigma(\mathrm{Sp}^{fp}, Ab)}_{\text{topological}}$$

<u>Q:</u> Which abelian categories can be described in this way?

<u>Def:</u> We say a homology theory $H : C \to A$ is <u>adapted</u> (<u>admits the ASS</u>) if:

1) A has enough injectives
2) for every injective $I \in A$, $\exists I_C \in C$ s.t.

   usually automatic.

$$[-, I_C] \simeq \mathrm{Hom}_A(H(-), I).$$

3) the canonical map $H(I_C) \to I$ is an iso.

   very much not automatic.

We get an ASS.

$$X := X_0 \hookrightarrow I_0 \to I_1 \to \cdots$$

$$I_0/X_0 = X_1 \hookrightarrow I_1, \quad I_1/X_1 = X_2 \hookrightarrow \cdots$$

H-Adams resolution.

H-mono into an H-injective.

of signature

$$\mathrm{Ext}_A^{s,t}(H(Y), H(X)) \Rightarrow [Y, X]_{t-s}.$$

<u>Ex:</u> $E_* : \mathrm{Sp} \to \mathrm{Comod}_{E_*E}$ where E is Adams-type

<u>Ex:</u> $\pi_* : \mathrm{Mod}(R) \to \mathrm{Mod}(R_*)$

<u>Ex:</u> (The universal homology thry).

$A(C) := \mathrm{Fun}(C^{op}, Ab)$ finitely presented.

C - stable, $A(C) := \mathrm{Fun}_{\Sigma}(C^{op}, Ab)$ (Freyd envelope)

$y : C \to A(C)$, $y(X)(Y) = [Y, X]$.

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[dr, "H"'] & A(C) \arrow[d, dotted, "\exists!"] \\ & \mathcal{A} \end{tikzcd}$$

$\exists!$ cocontinuous, exact functor of abelian cats.

<u>Thm:</u> Let $H : C \to \mathcal{A}$ be a homology theory such that $\mathcal{A}$ has enough injectives. Then, TFAE:

1) H is adapted
2) there exists a localizing subcat. $\mathcal{K} \subseteq A(C)$ s.t. $\mathcal{A} \simeq A(C)/\mathcal{K}$.
3) $A(C) \to \mathcal{A}$ admits a fully faithful right adjoint.

<u>Cor:</u> Adapted homology theories out of C form a poset.

(dual of the poset of localizing subcats. of $A(C)$).

lattice of ASSs.

Q: What does this correspondence look like?

<u>Def:</u> We say $\mathcal{M} \subseteq \mathrm{Fun}(\Delta^1, C)$ is a <u>mono class</u>:

1) eqv. are in $\mathcal{M}$
2) $f, g \in \mathcal{M} \Rightarrow f \circ g \in \mathcal{M}$, $f \circ g \in \mathcal{M} \Rightarrow g \in \mathcal{M}$
3) $\mathcal{M}$ is stable under pushouts
4) $\mathcal{M}$ is stable under sums.

We say $I \in C$ is $\mathcal{M}$-injective if it has RLP
... $\mathcal{M}$

wrt to maps in $\mathcal{M}$.

<u>Ex:</u> $H: C \to A$, $\mathcal{M} := H$-monic maps, $\mathcal{M}^{inj} :=$ objects of the form I_C.

<u>Ex:</u> R-ring spectrum
$\mathcal{M} := (R\otimes -)$-split injective maps   inclusion of a direct summand.
$\mathcal{M}^{inj} :=$ retracts of R-modules.

Adams resolution:

$$\begin{tikzcd} X \arrow[r] & R\otimes X \arrow[r] \arrow[d] & R\otimes \bar{R}\otimes X \arrow[r] & \cdots \\ & \bar{R}\otimes X \arrow[ur] & & \end{tikzcd}$$

In general different from the E_1-based ASS even when E is Adams-type.

Q: How to describe the E_2-term in this example?

<u>Thm:</u> There's a 1-1 correspondence between mono classes on C s.t. C has enough inj. and localizing subcats of $A(C)$.

<u>Cor:</u> Every mono class arises uniquely as the class of H-monic maps for an adapted homology thy H.

Every ASS has an E_2-page an Ext group in appropriate abelian category.

<u>Rmk:</u> In recent work, Balmer studies localizing ideals in $A(C)$ (<u>homological</u> <u>residue</u> <u>fields</u>).

<u>Notation:</u> We call fibres of $\mathcal{M}$-monic maps $\mathcal{M}$-zero
cofibres -||- $\mathcal{M}$-epic.

$$\cdots \to \Sigma^{-1}Z \xrightarrow[\text{M-zero}]{} X \xrightarrow{f \in M} Y \xrightarrow[\text{M-epic}]{} Z \xrightarrow[\text{M-zero}]{} \Sigma X \to \cdots$$

$$\begin{tikzcd}
\{ M \subseteq \mathrm{Fun}(\Delta^1, C) \mid M \text{ monic} \} \arrow[r, bend left, "A := Sh(C, M\text{-epic})^{fp}"] & \{ H : C \to A \mid \text{adapted} \} \arrow[l, bend left, "M := H\text{-monic maps.}"]
\end{tikzcd}$$

Fact: $Sh_{ep}(C) \subseteq P_{ep}(C)$.
this has an exact left adjoint when restricted to the subcats. of n-truncated objects for any fixed n.

Def: $Syn(C) := \varprojlim \tau_{\leq n} Sh_{ep}(C)$

Ex: $E_* : Sp_E \to Comod_{E_*E} \Rightarrow Syn(Sp) := (Syn_E)^{\wedge}$.

Ex: $\pi_* : Mod(R) \to Mod(R_*)$, $Syn(Mod(R)) := P_\Sigma(Mod(R)^{ff})$

$$\begin{tikzcd}
 & Syn(C) \arrow[d, "\pi_0"] \\
C \arrow[ur, "\nu_H"] \arrow[r, "H"'] & A.
\end{tikzcd}$$

where ν_H is a prestable enhancement to H.

Thm: The ∞-cat. $Syn(C)$ is the universal prestable enhancement of H, i.e. for any diagram

$$\begin{tikzcd}
 & & D \arrow[d, "\pi_0"] \\
C \arrow[urr, bend left] \arrow[r, "H"] & A \arrow[r] & D^{\heartsuit}
\end{tikzcd}$$

there's a unique $\mathrm{Syn}(\mathcal{C}) \to \mathcal{D}$ s.t. everything commutes.

$D(\mathcal{A})^{\mathrm{con}}$

$\mathrm{Syn}(\mathcal{C})$

universal property

universal prestable $\mathcal{D}$ with a functor $\mathcal{A} \to \mathcal{D}^{\heartsuit}$

universal prestable $\mathcal{D}$ with a functor $\mathcal{A} \xrightarrow{\cong} \mathcal{D}^{\heartsuit}$ + prestable enhancement of the composite Hof.

Franke's algebraicity

$$\begin{array}{ccccccccc}
 & & \mathrm{Syn}(\mathcal{C}) & \to \cdots & \cdots \to & \tau_{\leq 1}\mathrm{Syn}(\mathcal{C}) & \to & \tau_{\leq 0}\mathrm{Syn}(\mathcal{C}) \\
 & \nearrow & \cup\!\!\mid & & & \cup\!\!\mid & & \cup\!\!\mid \\
\mathcal{C} & \to & M_{\infty} & \to \cdots & \to & M_1 & \to & M_0
\end{array}$$

$\dim(\mathcal{A}) < \infty \Rightarrow$ there's a tp. range in which $h\mathcal{C} \simeq hM_k$.

Q: How do we show M_c are algebraic for low k?

Key: Need an appropriate functor $\beta^{\alpha} : \tau_{\leq k} D(\mathcal{A}) \to \tau_{\leq k}\mathrm{Syn}(\mathcal{C})$.

enough to define it on $\mathcal{A}$.

$$h_N \mathcal{C}^{\mathrm{inj}, e} \xrightarrow{\simeq} \mathcal{A}^{\mathrm{inj}, e} \Rightarrow$$

Bousfield splitting $\mathcal{A}^{\mathrm{inj}} \xrightarrow{\beta} h_N \mathcal{C}^{\mathrm{inj}}$. We put

$$\beta^{\alpha}(c^i) = (\tau_{\leq N} \beta(c^i)).$$

For general $a \in \mathcal{A}$, choose an inj. resolution $a \to c^{\bullet}$, put

$$\beta^{\alpha}(a) = \mathrm{Tot}(\beta^{\alpha}(c^{\bullet})).$$