

The Thom spectrum $MO\langle 4n \rangle$

Goal of this block: Study the first half (Sec 4 - Sec 10, Appendix B) of Burklund - Blahn - Senger; go through the proof of Thm 1.4 (Thm 1.1)

Recall the Kervaire - Milnor exact seq.

$$0 \rightarrow bP_{m+1} \rightarrow \Theta_m \rightarrow \text{coker}(J)_m = \pi_m \mathcal{S} / J$$

J-homomorphism

homotopy equiv. \hookrightarrow

group of closed, smooth, ori. manifolds $\cong \mathcal{S}^m$

m-homotopy spheres

m-homotopy spheres that are boundaries of parallelizable (m+1)-manifolds.

Thm 1.1 Let $k > 232$ and $0 \leq d \leq 3$ be integers. Suppose M is a $(k-1)$ -connected, almost closed $(2k+d)$ -manifold. Then the boundary $\partial M \in \Theta_{2k+d-1}$ has trivial image

$$0 = [\partial M] \in \text{coker}(J)_{2k+d-1}$$

In part, ∂M bounds a parallelizable manifold.

An important ingredient of proof of Thm 1.1 is the following, which is interesting on its own:

Thm 1.4 Let $MO\langle 4n \rangle$ denote the Thom spectrum of the canonical map

$$\tau_{\geq 4n} BO \rightarrow BO$$

For all $n > 31$, the unit map

$$\iota_*: \pi_{8n-1} \mathcal{S} \rightarrow \pi_{8n-1} MO\langle 4n \rangle$$

is surjective, and

$$\text{im}(\iota_*) = \text{im} \left(\pi_{8n-1} \mathcal{O} \xrightarrow{J} \pi_{8n-1} \mathcal{S} \right)$$

Plan for this block:

- Today: Study $MO\langle 4n \rangle$, the map i^* .
- 15.02: Reducing proof of 1.4 to the Toda bracket w in Jeremy's talk and prove Thm 1.4 using some blackbox like Thm 10.8
- 20.02: Prove the blackbox thms, e.g. Thm 10.8

§1. The Thom spectrum $MO\langle 4n \rangle$

Def (Thom sp. for a vector bundle) Let $\xi: E \rightarrow X$ be a vector bundle of rank n . The Thom spectrum $M\xi$ associated to ξ is the susp. spectrum $\Sigma^\infty \text{Th}(\xi)$.

In part: $M\xi_n = \Sigma^m \text{Th}(\xi) \approx \text{Th}(\underline{\mathbb{R}}^m \oplus \xi)$

Prop. Let $f: X \rightarrow BO(n)$ be a map classifying ξ , i.e.

$$\begin{array}{ccc} \xi_n \rightarrow EO(n) & & \\ \downarrow \cong & \searrow & \\ X \xrightarrow{f} BO(n) & & \end{array} \quad \text{with } \xi_n^X \times_{O(n)} \mathbb{R}^n \rightarrow X \text{ iso to } \xi$$

Then $\text{Th}(\xi) \approx \xi_{n+1} \wedge_{O(n)} S^n$

Furthermore we have

$$\begin{array}{ccc} \xi_{n+m} & \xrightarrow{\quad} & EO(n+m) \\ \downarrow \cong & & \downarrow \\ X \xrightarrow{f} BO(n) & \rightarrow \cdots \rightarrow & BO(n+m) \rightarrow \cdots \end{array}$$

with $\xi_{n+m}^X \times_{O(n)} \mathbb{R}^{n+m} \rightarrow X \cong \xi \oplus \underline{\mathbb{R}}^m$

$$\leadsto M\xi_n \cong (\xi_{n+m})_+ \wedge_{O(n)} S^n$$

Def. Given a map $X \xrightarrow{f} BO := \text{colim}_{n \geq 0} BO(n)$ of spaces,
 f is equivalent to the data of maps $f_n: X \rightarrow BO(n)$

for $n \geq 0$ s.t. $\begin{array}{ccc} X & & \\ f_n \downarrow & \searrow f_{n+1} & \\ BO(n) & \longrightarrow & BO(n+1) \end{array}$ commutes.

We define the Thom spectrum Mf associated to f
 as a sequential spectrum with

$$Mf_n := \text{Th}(\mathcal{E}_n \otimes_{O(n)} \mathbb{R}^n \rightarrow X) \\ \simeq (\mathcal{E}_n)_+ \wedge_{O(n)} S^n$$

where $\mathcal{E}_n \rightarrow X$ is the principal $O(n)$ bundle
 classified by f_n

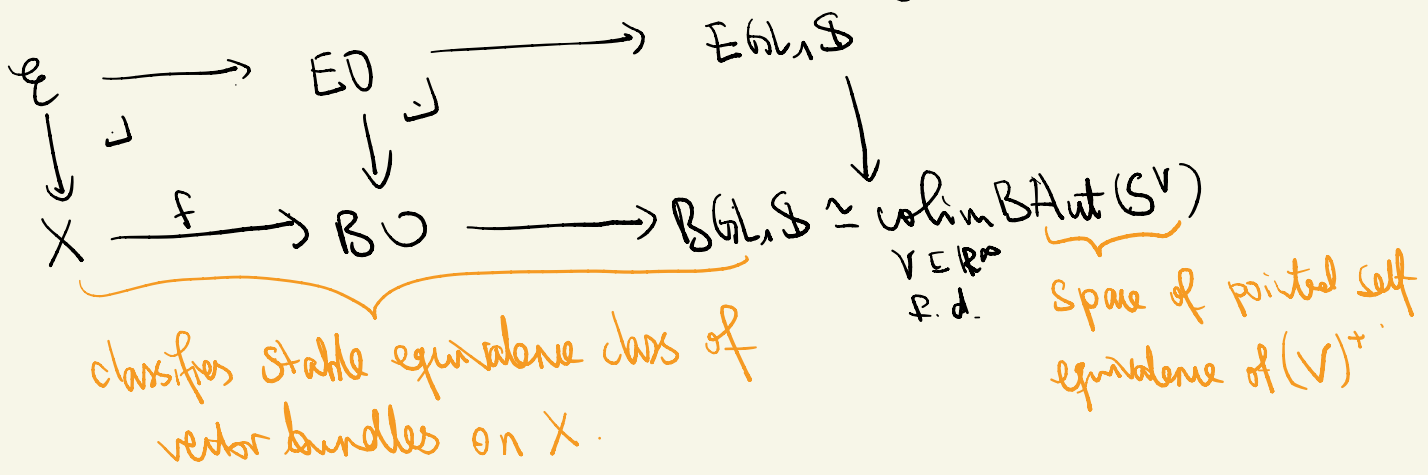
Def. Let $MO \langle 4n \rangle$ denote the Thom spectrum associated to the canonical map
 $\tau_{4n}: BO \rightarrow BO$.

Important remark • Given a map $f: X \rightarrow BO$, Mf is
 equivalently the Thom spectrum of

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & EO \\ \mathcal{E} \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & BO \end{array}$$

$$\begin{aligned} \simeq Mf &\simeq \Sigma^\infty \text{Th}(\mathcal{E}) \\ &\simeq \Sigma^\infty (\mathcal{E}_+ \wedge_{O} (\mathbb{R}^\infty)^+) \\ &\simeq \Sigma_+^\infty \mathcal{E}_+ \wedge_{\Sigma_+^\infty} S \end{aligned}$$

• we can extend the above pullback diagram to

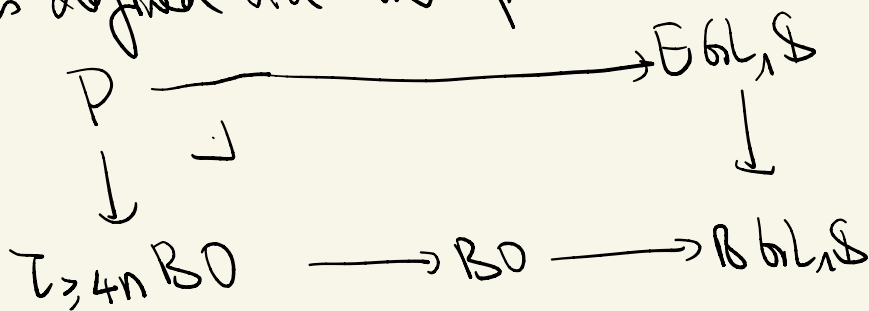


$$\begin{aligned}
 \leadsto Mf &\simeq \Sigma^\infty \mathrm{Th}(\mathcal{E} \rightarrow X) \simeq \Sigma^\infty (\mathcal{E}_+ \wedge_{B\mathrm{GL}_1\mathbb{S}} (\mathbb{R}^\infty)^+) \\
 &\simeq \Sigma_+^\infty \mathcal{E} \wedge_{\Sigma_+^\infty B\mathrm{GL}_1\mathbb{S}} \mathbb{S}
 \end{aligned}$$

Thus

$$M\mathcal{O}(4n) \simeq \Sigma_+^\infty P \wedge_{\Sigma_+^\infty B\mathrm{GL}_1\mathbb{S}} \mathbb{S}$$

where P is defined via the pullback



§§ $M\mathcal{O}(4n)$ as a particular Bar construction

Consider the map $\tau_{\geq 4n} B\mathcal{O} \rightarrow B\mathcal{O} \rightarrow B\mathrm{GL}_1\mathbb{S}$ defining $M\mathcal{O}(4n)$

Note that this is a map of infinite loop spaces, and thus equivalent to a map

$$\mathcal{E}: \tau_{\geq 4n} k_0 \rightarrow k_0 \rightarrow \mathrm{bgl}_1(\mathbb{S}) \text{ of connective spectra}$$

Consider the following diagram, denote $\mathrm{cgl}_1(\mathbb{S}) := \Sigma^{-1} \mathrm{bgl}_1(\mathbb{S})$

$$\begin{array}{ccccc}
 \Sigma^{-1} \tau_{\geq 4n} k_0 & \xrightarrow{\Sigma^{-1}} & \text{cyl}_1(\mathbb{S}) & \equiv & \text{cyl}_1(\mathbb{S}) \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \xrightarrow{\quad} & C\mathbb{S} & \xrightarrow{\quad} & e\text{-cyl}_1(\mathbb{S}) \simeq * \\
 & & \downarrow \perp & & \downarrow \\
 & & \tau_{\geq 4n} k_0 & \xrightarrow{\quad} & \text{br}_1(\mathbb{S})
 \end{array}$$

By definition, we know $MO\langle 4n \rangle$ fits in the

$$\begin{array}{ccccc}
 \Sigma_+^{\infty} \Omega^{\infty} \Sigma^{-1} \tau_{\geq 4n} k_0 & \xrightarrow{\quad} & \Sigma_+^{\infty} \Omega^{\infty} \text{cyl}_1(\mathbb{S}) & \xrightarrow{\quad} & \mathbb{S} \\
 \downarrow \varepsilon & & \downarrow \Gamma & & \downarrow h_{\Gamma} \\
 \mathbb{S} \simeq \Sigma_+^{\infty} \Omega^{\infty} * & \xrightarrow{\quad} & \Sigma_+^{\infty} P & \xrightarrow{\quad} & MO\langle 4n \rangle \\
 & & \text{" } \Sigma_+^{\infty} \Omega^{\infty} C\mathbb{S} \text{ " } & &
 \end{array}$$

$\leadsto MO\langle 4n \rangle \simeq \mathbb{S} \wedge_{\Sigma_+^{\infty} O\langle 4n-1 \rangle} \mathbb{S}$ where

- $O\langle 4n-1 \rangle := \Omega^{\infty} \Sigma^{-1} \tau_{\geq 4n} k_0$
- $\Sigma_+^{\infty} O\langle 4n-1 \rangle$ acts on the left \mathbb{S} component via the aug. map
 $\varepsilon: \Sigma_+^{\infty} O\langle 4n-1 \rangle \rightarrow \Sigma_+^{\infty} \Omega^{\infty} * = \mathbb{S}$
- $\Sigma_+^{\infty} O\langle 4n-1 \rangle$ acts on the right \mathbb{S} component via the map
 $J_+: \Sigma_+^{\infty} O\langle 4n-1 \rangle \rightarrow \Sigma_+^{\infty} \Omega^{\infty} \text{cyl}_1(\mathbb{S}) \rightarrow \mathbb{S}$

Def / Prop $MO\langle 4n \rangle$ is the geometric realization

$$| \text{Bar}(\mathbb{S}, \Sigma_+^{\infty} O\langle 4n-1 \rangle, \mathbb{S}) |$$

||

$$\left| \begin{array}{ccc}
 \xrightarrow{\quad} & \xrightarrow{\text{Id} \otimes J_+} & \xrightarrow{J_+} \\
 \xrightarrow{\quad} & \xrightarrow{m} & \xrightarrow{\varepsilon} \\
 \xrightarrow{\quad} & \xrightarrow{\varepsilon \otimes \text{Id}} & \xrightarrow{\quad} \\
 \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad}
 \end{array} \right. \Sigma_+^{\infty} O\langle 4n-1 \rangle^{\otimes 2} \xrightarrow{\quad} \Sigma_+^{\infty} O\langle 4n-1 \rangle \xrightarrow{\quad} \mathbb{S}$$

Recall that we want to prove

Thm 1.4 Let $MO\langle 4n \rangle$ denote the Thom spectrum of the canonical map

$$\tau_{\geq 4n} BO \rightarrow BO$$

For all $n \geq 1$, the unit map

$$i_*: \pi_{8n-1} \mathbb{S} \rightarrow \pi_{8n-1} MO\langle 4n \rangle$$

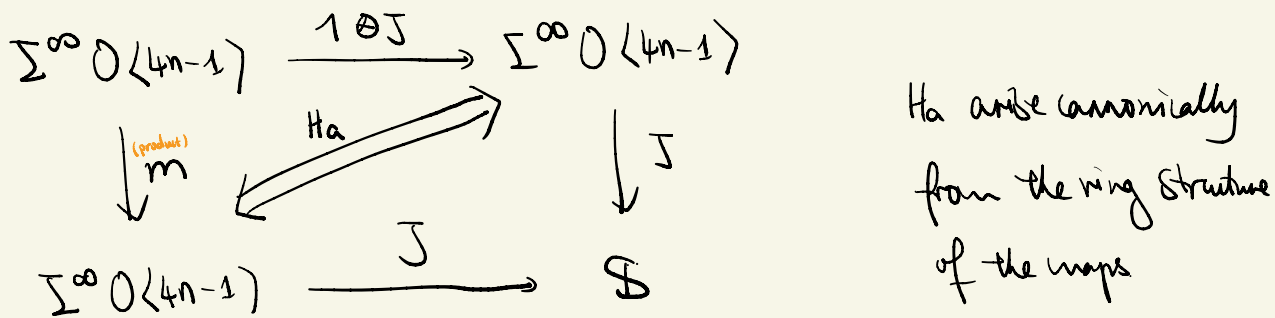
is surjective, and

$$\text{im}(i_*) = \text{im} \left(\pi_{8n-1} \mathbb{O} \xrightarrow{J} \pi_{8n-1} \mathbb{S} \right)$$

§ Replace i_* by another map

Not. Denote by J the non-unital E_∞ -ring map $\Sigma^\infty \mathbb{O}\langle 4n-1 \rangle \rightarrow \mathbb{S}$ induced by J_+ .

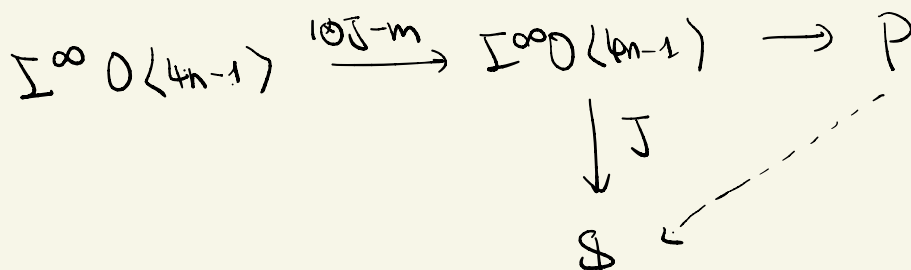
We have the following diagram of ring spectra up to homotopy



Remark We can view Ha as a nullhomotopy of $J \circ (1 \otimes J - m)$.

Not Denote $P := \text{cofib} \left(\Sigma^\infty \mathbb{O}\langle 4n-1 \rangle \xrightarrow{1 \otimes J - m} \Sigma^\infty \mathbb{O}\langle 4n-1 \rangle \right)$

By universal property of P , we have



Not. Denote $C := \text{cofib} (P \dashrightarrow \mathbb{S})$

Thm 5.2 There is an equivalence $\tau_{\leq 12n-2} C \simeq \tau_{\leq 12n-2} MO\langle 4n \rangle$ of spectra.

The unit map $\tau_{\leq 12n-2} S \rightarrow \tau_{\leq 12n-2} MO\langle 4n \rangle$ agrees with the natural map $\tau_{\leq 12n-2} S \rightarrow \tau_{\leq 12n-2} C$ from the cofibre sequence.

$$\text{Bar}(S, \Sigma_+^\infty O\langle 4n-1 \rangle, S)_{\leq 2}$$

Recall that

$$MO\langle 4n \rangle \simeq \left| \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \Sigma_+^\infty O\langle 4n-1 \rangle^{\otimes 2} \begin{array}{c} \xrightarrow{1 \otimes J_+} \\ \xrightarrow{m} \\ \xrightarrow{\varepsilon \circ 1} \end{array} \Sigma_+^\infty O\langle 4n-1 \rangle \begin{array}{c} \xrightarrow{J_+} \\ \xrightarrow{\varepsilon} \end{array} S \right|$$

Lemma 5.3 Let $X := \text{colim Bar}(S, \Sigma_+^\infty O\langle 4n-1 \rangle, S)_{\leq 2}$. Then

$$\tau_{\leq 12n-2} X \simeq \tau_{\leq 12n-2} MO\langle 4n \rangle$$

Pf: We have a tower

$$\begin{array}{ccccccc} S = \text{Bar}_0 & \rightarrow & \text{Bar}_{\leq 1} & \rightarrow & \text{Bar}_{\leq 2} & \rightarrow & \dots \rightarrow \text{Bar}_{\leq k} \rightarrow \dots \rightarrow MO\langle 4n \rangle \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Sigma^{0+1} O\langle 4n-1 \rangle & & \Sigma^{0+2} O\langle 4n-1 \rangle^{\otimes 2} & & \Sigma^{k+\infty} O\langle 4n-1 \rangle^{\otimes k} \end{array}$$

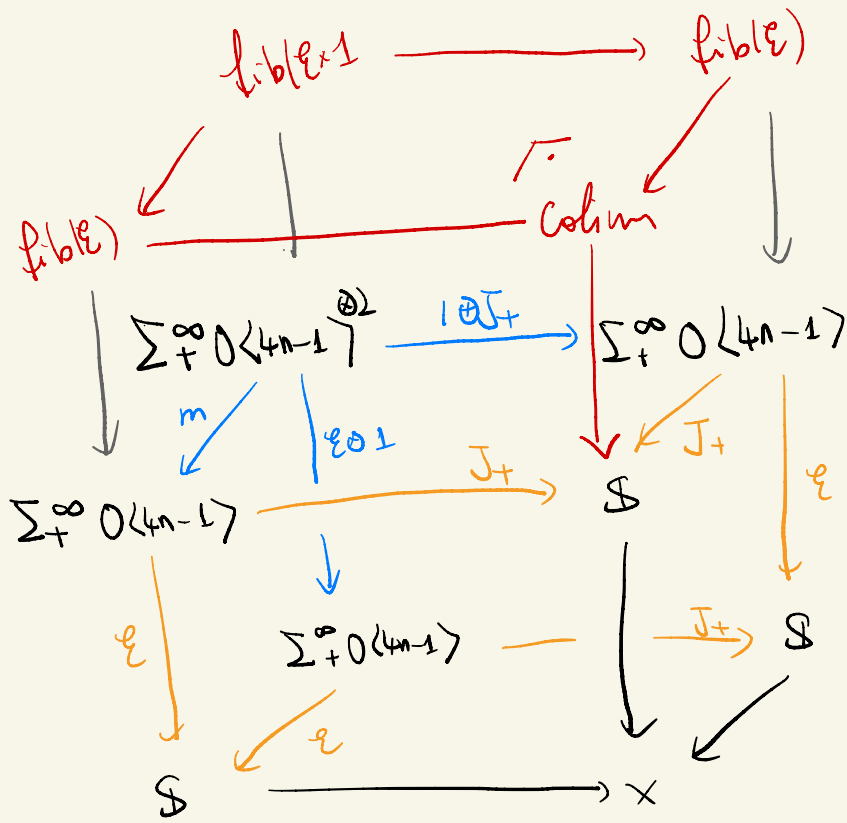
cofibres

and for $k \geq 3$, $\Sigma^k \Sigma^\infty O\langle 4n-1 \rangle^{\otimes k}$ is $12n$ -connective. □

Proof of Thm 5.2 It suffices if we proved $\tau_{\leq 12n-2} X \simeq \tau_{\leq 12n-2} C$!

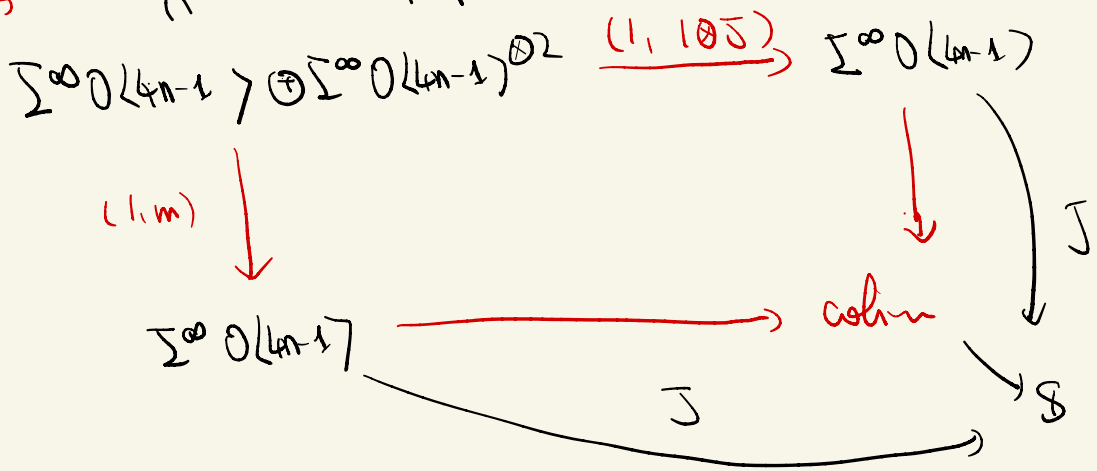
We can present X in the following coartesian cube:

$$\begin{array}{ccccc} & & \Sigma_+^\infty O\langle 4n-1 \rangle^{\otimes 2} & \xrightarrow{1 \otimes J_+} & \Sigma_+^\infty O\langle 4n-1 \rangle \\ & \swarrow m & \downarrow \varepsilon \circ 1 & \searrow J_+ & \swarrow J_+ \\ \Sigma_+^\infty O\langle 4n-1 \rangle & \xrightarrow{J_+} & S & \xrightarrow{J_+} & S \\ \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \varepsilon \\ S & \xrightarrow{J_+} & \Sigma_+^\infty O\langle 4n-1 \rangle & \xrightarrow{J_+} & S \\ \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \varepsilon \\ S & \xrightarrow{J_+} & S & \xrightarrow{J_+} & S \\ & & & & \downarrow \varepsilon \\ & & & & X \end{array}$$



X is the cofibre of the map $\text{colim} \rightarrow S$

The $\begin{matrix} \leftarrow & \rightarrow \\ \rightarrow & \leftarrow \end{matrix}$ diagram simplifies to



Thus colim is equivalent to the cofibre of the map

$$\Sigma_+^{\infty} O(4n-1) \otimes 2 \xrightarrow{1 \otimes J - m} \Sigma_+^{\infty} O(4n-1),$$

which is P. Thus $X \simeq \text{cofib}(P \rightarrow S) = C$

As for the unit map $S \rightarrow MO(4n)$, which is

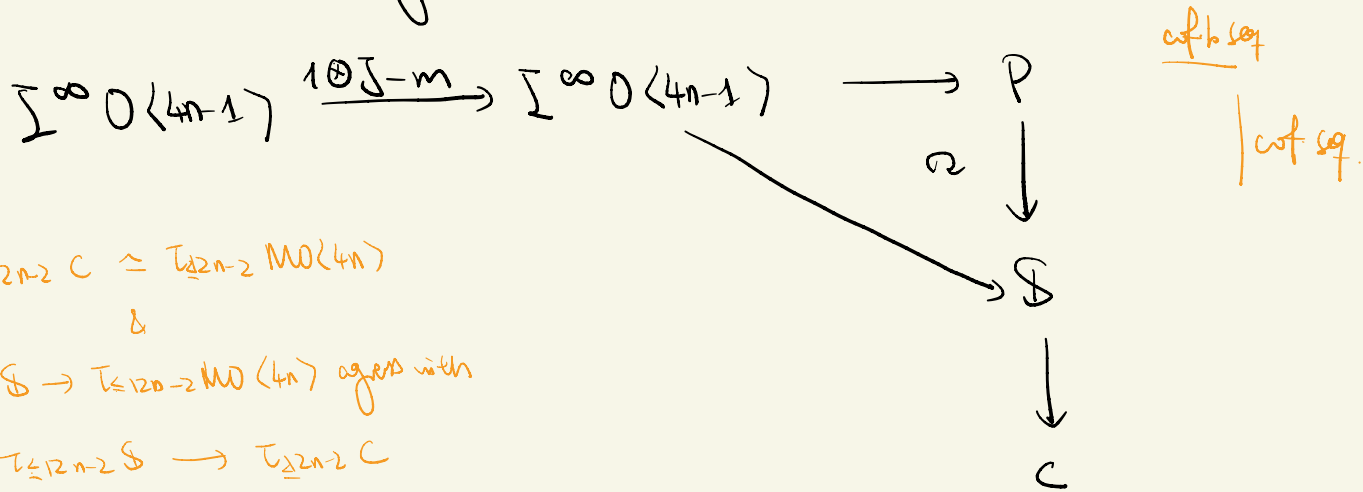
$$S \rightarrow \text{Bar}_{\leq 1} \rightarrow \dots \rightarrow MO(4n)$$

factors through $X := \text{colim Bar}_2$ and fits well in the cube. \square

Summarised; from the diagram $\Sigma^\infty O\langle 4n-1 \rangle \xrightarrow{1 \otimes J} \Sigma^\infty O\langle 4n-1 \rangle$

$$\begin{array}{ccc}
 \Sigma^\infty O\langle 4n-1 \rangle & \xrightarrow{1 \otimes J} & \Sigma^\infty O\langle 4n-1 \rangle \\
 \downarrow \text{(product)} & \nearrow Ha & \downarrow J \\
 \Sigma^\infty O\langle 4n-1 \rangle & \xrightarrow{J} & \mathcal{S}
 \end{array}$$

We obtain another diagram:



St. $T_{\leq 12n-2} C \cong T_{\leq 2n-2} MO\langle 4n \rangle$

$T_{\leq 12n-2} \mathcal{S} \rightarrow T_{\leq 12n-2} MO\langle 4n \rangle$ agrees with

$T_{\leq 12n-2} \mathcal{S} \rightarrow T_{\geq 2n-2} C$

Consider the l.e.s of π_* of $P \rightarrow \mathcal{S} \rightarrow C$ and "replace" C by $MO\langle 4n \rangle$

$$\pi_{8n-1}(P) \rightarrow \pi_{8n-1}(\mathcal{S}) \rightarrow \pi_{8n-1}(MO\langle 4n \rangle) \rightarrow \pi_{8n-2}(P) \rightarrow \pi_{8n-2}(\mathcal{S}).$$

§ Homotopy of $\Sigma^\infty O(4n-1)$

Not Denote by $x \in \pi_{4n-1}(\Sigma^\infty O(4n-1))$ a generator.

Since $\Sigma^\infty O(4n-1)$ is non-unital E_∞ -ring, we have

$$x^2 \in \pi_{8n-2}(\Sigma^\infty O(4n-1))$$

Also, denote by $J(x) \in \pi_{4n-1}(\mathcal{S})$ the composite

$$\mathcal{S}^{4n-1} \xrightarrow{x} \Sigma^\infty O(4n-1) \xrightarrow{J} \mathcal{S}$$

Cor 4.8 i) $\pi_{8n-2} \Sigma^\infty O(4n-1) \cong \mathbb{Z}/2\mathbb{Z} \langle x^2 \rangle$

ii) $\pi_{8n-1} \Sigma^\infty O(4n-1) \cong \pi_{8n} k_0 \cong \mathbb{Z}$

Lemma 4.10 The element $xJ(x) \in \pi_{8n-2} \Sigma^\infty O(4n-1)$ is zero

Thm 4.11 For $4n-1 \leq l \leq 8n-1$, the image of

$$\pi_l(\Sigma^\infty O(4n-1)) \xrightarrow{J} \pi_l(\mathcal{S})$$

$$\text{is } \mathcal{I}_l = \text{im}(\pi_l(O) \xrightarrow{J\text{-homo}} \pi_l(\mathcal{S}))$$

Key ingredient for the proofs: Goodwillie calculus of the id of augmented E_∞ -ring spectra.

Lemma 4.5 R Eoo-ring spectra with aug. $\mathcal{E}: R \rightarrow \mathcal{S}$. S.t. $\text{fib}(\mathcal{E})$ is 0-connected. The Goodwillie tower of id evaluated at R gives a convergent tower of Eoo-ring spectra:

$$\begin{array}{ccccccc}
 D_n(\text{TAQ}(R, \mathcal{S})) & & D_2(\text{TAQ}(R, \mathcal{S})) & \text{TAQ}(R, \mathcal{S}) & \mathcal{S} \\
 \downarrow & & \downarrow & \downarrow & \downarrow \simeq \\
 R \rightarrow \dots \rightarrow P_n(R) \rightarrow \dots \rightarrow P_2(R) \rightarrow P_1(R) \rightarrow P_0(R)
 \end{array}$$

S.t. $R \rightarrow P_0(R)$ is the augmentation \mathcal{E} . Here

$$D_n(\text{TAQ}(R, \mathcal{S})) := (\text{TAQ}(R, \mathcal{S})^{\otimes n})_{h\Sigma_n}$$

Cor. 4.6 We have convergent tower of spectra

$$\begin{array}{ccccccc}
 D_n(\Sigma^{-1} \tau_{\geq 4n} k_0) & & D_3(\dots) & D_2(\dots) & \Sigma^{-1} \tau_{\geq 4n} k_0 \\
 \downarrow & & \downarrow & \downarrow & \downarrow \simeq \\
 \Sigma^{\infty} O\langle 4n-1 \rangle \rightarrow \dots \rightarrow Q_n \rightarrow \dots \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1 & (*)
 \end{array}$$

PF Take $R = \Sigma^{\infty}_+ O\langle 4n-1 \rangle$ with augmentation \mathcal{E} in the diagram in Lem 4.5

Fact: $\text{TAQ}(R, \mathcal{S}) \simeq \Sigma^{-1} \tau_{\geq 4n} k_0$

Note that $\Sigma^{\infty} O\langle 4n-1 \rangle \simeq \text{fib}(\mathcal{E})$ and set

$$Q_n := \text{fib}(P_n(R) \rightarrow P_{n-1}(R))$$

Lemma 4.7 - $\pi_{8n-2} D_2(\Sigma^{-1} \tau_{\geq 4n} k_0) \cong \mathbb{Z}/2\mathbb{Z}$

$\pi_{8n-1} D_2(\Sigma^{-1} \tau_{\geq 4n} k_0) \cong 0$

- The generator of $\pi_{8n-2} D_2(\Sigma^{-1} \tau_{\geq 4n} k_0)$ survives in the SS associated to (*) and detects $X^2 \in \pi_{8n-2} \Sigma^\infty O\langle 4n-1 \rangle$.

Cor 4.8 i) $\pi_{8n-2} \Sigma^\infty O\langle 4n-1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \{X^2\}$

ii) $\pi_{8n-1} \Sigma^\infty O\langle 4n-1 \rangle \cong \pi_{8n} k_0 \cong \mathbb{Z}$

Sketch - $\pi_{\leq 8n-1} D_k(\Sigma^{-1} \tau_{\geq 4n} k_0) \simeq *$ for $k > 2$

\leadsto a l.e.s. on π_* from (*):

$$0 \rightarrow \pi_{8n-1}(\Sigma^\infty O\langle 4n-1 \rangle) \rightarrow \pi_{8n-1}(\Sigma^{-1} \tau_{\geq 4n} k_0) \rightarrow \pi_{8n-2}(D_2(\Sigma^{-1} \tau_{\geq 4n} k_0)) \xrightarrow{\downarrow} \pi_{8n-2}(\Sigma^\infty O\langle 4n-1 \rangle) \xleftarrow{\cong} \pi_{8n-2}(\Sigma^{-1} \tau_{\geq 4n} k_0)$$

$\mathbb{Z}/2\mathbb{Z}$
 \cong

Claim $\pi_j(\Sigma^\infty O\langle 4n-1 \rangle) \rightarrow \pi_j(\Sigma^{-1} \tau_{\geq 4n} k_0)$ are surjective

Sketch $\Sigma^\infty O\langle 4n-1 \rangle \rightarrow \Sigma^{-1} \tau_{\geq 4n} k_0$ is adjoint to

id: $O\langle 4n-1 \rangle \rightarrow \Omega^{\infty+1} \tau_{\geq 4n} k_0$

$\leadsto \Omega^{\infty+1} \tau_{\geq 4n} k_0 \rightarrow \Omega^\infty \Sigma^\infty O\langle 4n-1 \rangle \rightarrow \Omega^{\infty+1} \tau_{\geq 4n} k_0$
 $\simeq \text{id } \Sigma^{-1} \tau_{\geq 4n} k_0$ □ claim

\leadsto Cor. follows by examining the l.e.s.

Lemma 4.10

The element $\alpha J(x) \in \pi_{8n-2} \Sigma^{\infty} O(4n-1)$ is zero

Sketch

Assume $\alpha J(x) \neq 0$, then $\alpha J(x) = x^2$, under the identification $\pi_{8n-2} \Sigma^{\infty} O(4n-1) \cong \mathbb{Z}/2\mathbb{Z}\{x^2\}$

Prop 10.22: x^2 has HF_2 -Adams filtration 1.

Recall that α is a generator of $\pi_{4n-1} \Sigma^{\infty} O(4n-1)$, and thus the suspension of an unstable class in $\pi_{4n-1} O(4n-1)$.

So $J(x)$ is in the image J_{4n-1} of J .

\leadsto As $n > 3$, $J(x)$ has HF_2 -Adams filtration 3 and $\alpha J(x)$ has at least HF_2 -Adams filtration of $J(x)$ \geq □

Thm 4.11

For $4n-1 \leq l \leq 8n-1$, the image of

$$\pi_l(\Sigma^{\infty} O(4n-1)) \xrightarrow{J} \pi_l(\mathcal{S})$$

$$\text{is } \mathcal{J}_l = \text{im}(\pi_l(O) \xrightarrow{J\text{-hom}} \pi_l(\mathcal{S}))$$

Sketch It suffices to show that in this range, every

$y \in \pi_l(\Sigma^{\infty} O(4n-1))$ is the suspension of an unstable class.

Take a look at the Goodwillie tower (*) for $\Sigma^{\infty} O(4n-1)$

$$\begin{array}{ccccccc}
 & & D_n(\Sigma^{-1}\tau_{\geq 4n}k_0) & & D_3(\dots) & & D_2(\dots) & & \Sigma^{-1}\tau_{\geq 4n}k_0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \cong \\
 \Sigma^\infty \mathbb{O}\langle 4n-1 \rangle & \rightarrow \dots \rightarrow & Q_n & \rightarrow \dots \rightarrow & Q_3 & \rightarrow & Q_2 & \rightarrow & Q_1 \quad (*)
 \end{array}$$

- for $4n-1 \leq l \leq 8n-3$, $\pi_l(D_k(\Sigma^{-1}\tau_{\geq 4n}k_0)) \cong 0$, $k \geq 2$

$$\leadsto \pi_l(\Sigma^\infty \mathbb{O}\langle 4n-1 \rangle) \cong \pi_l(\Sigma^{-1}\tau_{\geq 4n}k_0) \quad \checkmark$$

- for $l = 8n-1$, Cor 4.8 $\pi_{8n-1}(\Sigma^\infty \mathbb{O}\langle 4n-1 \rangle) \cong \pi_{8n}k_0 \quad \checkmark$

- for $l = 8n-2$, $\pi_{8n-2}(\Sigma^\infty \mathbb{O}\langle 4n-1 \rangle) \cong \mathbb{Z}/2\mathbb{Z}\langle x^2 \rangle$

Prop. 15.11 $J(x^2) = J(x)^2 = 0$

□