The Thom spectrum $MO\langle 4n\rangle$

Goal of this block: Study the first half (Sec 4 – Sec 10, Appendix B) of Burkland – Hahn – Senger; go through the proof of Thm 1.4 (Thm 1.1)

Recall the Kervaire – Milnor exact seq.

$$0 \to bP_{m+1} \to \Theta_m \to \mathrm{coker}(J)_m = \pi_m \mathbb{S}/J$$

J-homomorphism

bP_{m+1}: m-homotopy spheres that are boundaries of parallelizable (m+1)-manifolds.

Θ_m: group of $\underbrace{\text{closed, smooth, ori. manifolds} \overset{\text{htpy equiv.}}{\simeq} S^m}_{m\text{-homotopy spheres}}$

Thm 1.1 Let $k > 232$ and $0 \leq d \leq 3$ be integers. Suppose M is a $(k-1)$-connected, almost closed $(2k+d)$-manifold. Then the boundary $\partial M \in \Theta_{2k+d-1}$ has trivial image

$$0 = [\partial M] \in \mathrm{coker}(J)_{2k+d-1}$$

In part. ∂M bounds a parallelizable manifold.

An important ingredient of proof of Thm 1.1 is the following, which is interesting on its own:

Thm 1.4 Let $MO\langle 4n\rangle$ denote the Thom spectrum of the canonical map

$$\tau_{\geq 4n} BO \longrightarrow BO$$

For all $n > 31$, the unit map

$$\iota_*: \pi_{8n-1}\mathbb{S} \longrightarrow \pi_{8n-1} MO\langle 4n\rangle$$

is surjective, and

$$\mathrm{im}(\iota_*) = \mathrm{im}\left(\pi_{8n-1} O \xrightarrow{J} \pi_{8n-1}\mathbb{S}\right)$$

Plan for this block:

- Today: Study $MO\langle 4n\rangle$, the map i_*.
- 15.02. : Reducing proof of 1.4 to the Toda brack w in Jeremy's talk and prove Thm 1.4 using some black box like Thm 10.8
- 22.02 : Prove the blackbox thms, e.g. Thm 10.8

§1. The Thom spectrum $MO\langle 4n\rangle$

Def (Thom sp. for a vector bundle) Let $\xi : E \to X$ be a vector bundle of rank n. The Thom spectrum $M\xi$ associated to ξ is the susp. spectrum $\Sigma^\infty Th(\xi)$.

In part. $M\xi_n = \Sigma^m Th(\xi) \approx Th(\underline{\mathbb{R}}^m \oplus \xi)$

Rmk. Let $f : X \to BO(n)$ be a map classifying ξ, i.e.

$$\begin{tikzcd}
\mathcal{E}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EO(n) \arrow[d] \\
X \arrow[r, "f"'] & BO(n)
\end{tikzcd}$$

with $\mathcal{E}_n \times_{O(n)} \mathbb{R}^n \to X$ iso to ξ.

Then $Th(\xi) \simeq \mathcal{E}_{n+} \wedge_{O(n)} S^n$

Furthermore we have

$$\begin{tikzcd}
\mathcal{E}_{n+m} \arrow[rrrr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & & & EO(n+m) \arrow[d] \\
X \arrow[r, "f"'] & BO(n) \arrow[r] & \cdots \arrow[r] & & BO(n+m) \arrow[r] & \cdots
\end{tikzcd}$$

with $\mathcal{E}_{n+m} \times_{O(n)} \mathbb{R}^{n+m} \to X \cong \xi \oplus \underline{\mathbb{R}}^m$

$\rightsquigarrow M\xi_n \cong (\mathcal{E}_{n+m})_+ \wedge_{O(n)} S^n$

Def. Given a map $X \xrightarrow{f} BO := \operatorname{colim}_{n \geq 0} BO(n)$ of spaces, f is equivalent to the data of maps $f_n : X \to BO(n)$ for $n \geq 0$ s.t.

$$\begin{tikzcd} X \arrow[d, "f_n"'] \arrow[dr, "f_{n+1}"] \arrow[d, phantom, shift left=4, "\circlearrowleft"] & \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

commutes.

We define the Thom spectrum Mf associated to f as a sequential spectrum with

$$Mf_n := Th(\mathcal{E}_n \otimes_{O(n)} \mathbb{R}^n \to X)$$
$$\simeq (\mathcal{E}_n)_+ \wedge_{O(n)} S^n$$

where $\mathcal{E}_n \to X$ is the principal $O(n)$ bundle classified by f_n

Def. Let $MO\langle 4n \rangle$ denote the Thom spectrum associated to the canonical map $\tau_{\geq 4n} BO \to BO$.

Important remark • Given a map $f : X \to BO$, Mf is equivalently the Thom spectrum of

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "\xi"'] \arrow[dr, phantom, "\lrcorner", very near start] & EO \arrow[d] \\ X \arrow[r, "f"'] & BO \end{tikzcd}$$

$\rightsquigarrow Mf \simeq \Sigma^\infty Th(\xi)$
$\simeq \Sigma^\infty (\mathcal{E}_+ \wedge_O (\mathbb{R}^\infty)^+)$
$\simeq \Sigma^\infty_+ \mathcal{E}_+ \wedge_{\Sigma^\infty_+ O} \mathbb{S}$

- we can extend the above pullback diagram to

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EO \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EGL_1\mathbb{S} \arrow[d] \\
X \arrow[r, "f"] & BO \arrow[r] & BGL_1\mathbb{S} \simeq \operatorname*{colim}_{\substack{V\subseteq\mathbb{R}^\infty \\ \text{f.d.}}} B\underbrace{\mathrm{Aut}(S^V)}_{\text{space of pointed self equivalence of } (V)^+}
\end{tikzcd}$$

$$\underbrace{X \xrightarrow{f} BO \longrightarrow BGL_1\mathbb{S}}_{\text{classifies stable equivalence class of vector bundles on } X.}$$

$$\rightsquigarrow Mf \simeq \Sigma^\infty Th(\mathcal{E}\to X) \simeq \Sigma^\infty\left(\mathcal{E}_+ \wedge_{GL_1\mathbb{S}} (\mathbb{R}^\infty)^+\right)$$

$$\simeq \Sigma^\infty_+ \mathcal{E} \wedge_{\Sigma^\infty_+ GL_1\mathbb{S}} \mathbb{S}$$

Thus

$$MO\langle 4n\rangle \simeq \Sigma^\infty_+ P \wedge_{\Sigma^\infty_+ GL_1\mathbb{S}} \mathbb{S}$$

where P is defined via the pullback

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & EGL_1\mathbb{S} \arrow[d] \\
\tau_{\geq 4n}BO \arrow[r] & BO \arrow[r] & BGL_1\mathbb{S}
\end{tikzcd}$$

§3 $MO\langle 4n\rangle$ as a particular Bar construction

Consider the map $\tau_{\geq 4n}BO \to BO \longrightarrow BGL_1\mathbb{S}$ defining $MO\langle 4n\rangle$
Note that this is a map of infinite loop spaces, and thus equivalent to a map

$$\xi : \tau_{\geq 4n}ko \longrightarrow ko \longrightarrow bgl_1(\mathbb{S})$$ of connective spectra

Consider the following diagram, denote $gl_1(\mathbb{S}) := \Sigma^{-1} bgl_1(\mathbb{S})$

$$\begin{tikzcd}
\Sigma^{-1}\tau_{\geq 4n}ko \arrow[r, "\Sigma^{-1}\xi"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & gl_1(\mathbb{S}) \arrow[r, equal] \arrow[d] & gl_1(\mathbb{S}) \arrow[d] \\
* \arrow[r] & C\xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & e\,gl_1(\mathbb{S}) \simeq * \arrow[d] \\
 & \tau_{\geq 4n}ko \arrow[r] & b\,gl_1(\mathbb{S})
\end{tikzcd}$$

By definition, we know $MO\langle 4n\rangle$ fits in the

$$\begin{tikzcd}
 & \Sigma^\infty_+\Omega^\infty gl_1(\mathbb{S}) \arrow[d, phantom, "\shortparallel"] & \\
\Sigma^\infty_+\Omega^\infty\Sigma^{-1}\tau_{\geq 4n}ko \arrow[r] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\ulcorner"] & \Sigma^\infty_+ GL_1(\mathbb{S}) \arrow[r] \arrow[d] \arrow[dr, phantom, "h\ulcorner"] & \mathbb{S} \arrow[d] \\
\mathbb{S} \simeq \Sigma^\infty_+\Omega^\infty * \arrow[r] & \Sigma^\infty_+ P \arrow[r] \arrow[d, phantom, "\shortparallel"] & MO\langle 4n\rangle \\
 & \Sigma^\infty_+\Omega^\infty C\xi &
\end{tikzcd}$$

$\rightsquigarrow$ $MO\langle 4n\rangle \simeq \mathbb{S} \wedge_{\Sigma^\infty_+ O\langle 4n-1\rangle} \mathbb{S}$ where

- $O\langle 4n-1\rangle := \Omega^\infty\Sigma^{-1}\tau_{\geq 4n}ko$
- $\Sigma^\infty_+ O\langle 4n-1\rangle$ acts on the left $\mathbb{S}$ component via the aug. map
  $\varepsilon : \Sigma^\infty_+ O\langle 4n-1\rangle \to \Sigma^\infty_+\Omega^\infty * = \mathbb{S}$
- $\Sigma^\infty_+ O\langle 4n-1\rangle$ acts on the right $\mathbb{S}$ component via the map
  $J_+ : \Sigma^\infty_+ O\langle 4n-1\rangle \to \Sigma^\infty_+ GL_1(\mathbb{S}) \to \mathbb{S}$

__Def / Prop__ $MO\langle 4n\rangle$ is the geometric realisation

$$\left| Bar(\mathbb{S}, \Sigma^\infty_+ O\langle 4n-1\rangle, \mathbb{S}) \right|$$

$$\shortparallel$$

$$\left| \begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \arrow[r, shift left=3, "1\otimes J_+"] \arrow[r, "m"] \arrow[r, shift right=3, "\varepsilon\otimes 1"'] & \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[r, shift left=2, "J_+"] \arrow[r, shift right=2, "\varepsilon"'] & \mathbb{S}
\end{tikzcd} \right|$$

Recall that we want to prove

> <u>Thm 1.4</u> Let $MO\langle 4n\rangle$ denote the Thom spectrum of the canonical map
>
> $$\tau_{\geq 4n} BO \longrightarrow BO$$
>
> For all $n > 31$, the unit map
>
> $$\iota_* : \pi_{8n-1}\mathbb{S} \longrightarrow \pi_{8n-1} MO\langle 4n\rangle$$
>
> is surjective, and
>
> $$\mathrm{im}(\iota_*) = \mathrm{im}\left(\pi_{8n-1} O \xrightarrow{J} \pi_{8n-1}\mathbb{S}\right)$$

§ Replace i_* by another map

<u>Not.</u> Denote by J the non-unital E_∞-ring map $\Sigma^\infty O\langle 4n-1\rangle \to \mathbb{S}$ induced by J_+.

We have the following diagram of ring spectra up to homotopy

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "1\otimes J"] \arrow[d, "\overset{(\text{product})}{m}"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J"] \arrow[dl, Rightarrow, "Ha"'] \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "J"] & \mathbb{S}
\end{tikzcd}$$

Ha arise canonically from the ring structure of the maps

<u>Rmk</u> We can view Ha as a nullhomotopy of $J \circ (1\otimes J - m)$.

<u>Not</u> Denote $P := \mathrm{cofib}\left(\Sigma^\infty O\langle 4n-1\rangle \xrightarrow{1\otimes J - m} \Sigma^\infty O\langle 4n-1\rangle\right)$

By universal property of P, we have

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "1\otimes J - m"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[r] \arrow[d, "J"] & P \arrow[dl, dashed] \\
& \mathbb{S} &
\end{tikzcd}$$

<u>Not.</u> Denote $C := \mathrm{cofib}\,(P \dashrightarrow \mathbb{S})$

Thm 5.2 There is an equivalence $\tau_{\le 12n-2} C \simeq \tau_{\le 12n-2} MO\langle 4n\rangle$ of spectra. The unit map $\tau_{\le 12n-2}\mathbb{S} \longrightarrow \tau_{\le 12n-2} MO\langle 4n\rangle$ agrees with the natural map $\tau_{\le 12n-2}\mathbb{S} \longrightarrow \tau_{\le 12n-2} C$ from the cofibre sequence.

Recall that

$$MO\langle 4n\rangle \simeq \left|\ \cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \overbrace{\Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \overset{\xrightarrow{1\otimes J_+}}{\underset{\xrightarrow[\varepsilon\otimes 1]{}}{\xrightarrow{m}}} \Sigma^\infty_+ O\langle 4n-1\rangle \overset{\xrightarrow{J_+}}{\xrightarrow[\varepsilon]{}} \mathbb{S}}^{\mathrm{Bar}(\mathbb{S},\, \Sigma^\infty_+ O\langle 4n-1\rangle,\, \mathbb{S})_{\le 2}} \right|$$

Lemma 5.3 Let $X := \mathrm{colim}\, \mathrm{Bar}(\mathbb{S}, \Sigma^\infty_+ O\langle 4n-1\rangle, \mathbb{S})_{\le 2}$. Then

$$\tau_{\le 12n-2} X \simeq \tau_{\le 12n-2} MO\langle 4n\rangle$$

Pf: We have a tower

$$\begin{tikzcd}
\mathbb{S} = \mathrm{Bar}_0 \arrow[r] & \mathrm{Bar}_{\le 1} \arrow[r] \arrow[d] & \mathrm{Bar}_{\le 2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{Bar}_{\le k} \arrow[r] \arrow[d] & \cdots \arrow[r] & MO\langle 4n\rangle \\
\text{cofibres} & \Sigma^{\infty+1} O\langle 4n-1\rangle & \Sigma^{\infty+2} O\langle 4n-1\rangle^{\otimes 2} & & \Sigma^{k+\infty} O\langle 4n-1\rangle^{\otimes k} & &
\end{tikzcd}$$

and for $k \geq 3$, $\Sigma^k \Sigma^\infty O\langle 4n-1\rangle^{\otimes k}$ is $12n$-connective. $\square$

Proof of Thm 5.2 It suffices if we proved $\tau_{\le 12n-2} X \simeq \tau_{\le 12n-2} C$!

We can present X in the following cocartesian cube:

$$\Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \overset{\xrightarrow{1\otimes J_+}}{\underset{\xrightarrow[\varepsilon\otimes 1]{}}{\xrightarrow{m}}} \Sigma^\infty_+ O\langle 4n-1\rangle \overset{\xrightarrow{J_+}}{\xrightarrow[\varepsilon]{}} \mathbb{S}$$

$$\begin{tikzcd}
 & \Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \arrow[rr, "1\otimes J_+"] \arrow[dl, "m"'] \arrow[dd, "\varepsilon\otimes 1"] & & \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[dl, "J_+"'] \arrow[dd, "\varepsilon"] \\
\Sigma^\infty_+ O\langle 4n-1\rangle \arrow[rr, "J_+"] \arrow[dd, "\varepsilon"'] & & \mathbb{S} \arrow[dd] & \\
 & \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[rr, "J_+"] \arrow[dl, "\varepsilon"] & & \mathbb{S} \arrow[dl] \\
\mathbb{S} \arrow[rr] & & X &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{fib}(\varepsilon\times 1 \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{fib}(\varepsilon) \arrow[dl] \arrow[dd] \\
\mathrm{fib}(\varepsilon) \arrow[rr] \arrow[dd] & & \mathrm{colim} \arrow[dd] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & \Sigma^\infty_+ O\langle 4n-1\rangle^{\otimes 2} \arrow[rr, "1\otimes J_+"] \arrow[dl, "m"'] \arrow[dd, "\varepsilon\otimes 1"] & & \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[dl, "J_+"] \arrow[dd, "\varepsilon"] \\
\Sigma^\infty_+ O\langle 4n-1\rangle \arrow[rr, "J_+"] \arrow[dd, "\varepsilon"'] & & \mathbb{S} \arrow[dd] & \\
 & \Sigma^\infty_+ O\langle 4n-1\rangle \arrow[rr, "J_+"] \arrow[dl, "\varepsilon"] & & \mathbb{S} \arrow[dl] \\
\mathbb{S} \arrow[rr] & & X &
\end{tikzcd}$$

X is the cofibre of the map colim → $\mathbb{S}$

The $\square$ diagram simplifies to

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle \oplus \Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \arrow[r, "(1,\, 1\otimes J)"] \arrow[d, "(1,m)"'] & \Sigma^\infty O\langle 4n-1\rangle \arrow[d] \arrow[ddr, bend left, "J"] & \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r] \arrow[drr, bend right, "J"'] & \mathrm{colim} \arrow[dr] & \\
 & & \mathbb{S}
\end{tikzcd}$$

Thus colim is equivalent to the cofibre of the map

$$\Sigma^\infty O\langle 4n-1\rangle^{\otimes 2} \xrightarrow{1\otimes J - m} \Sigma^\infty O\langle 4n-1\rangle,$$

which is P. Thus $X \simeq \mathrm{cofib}(P \to \mathbb{S}) = C$

As for the unit map $\mathbb{S} \to MO\langle 4n\rangle$, which is

$$\mathbb{S} \to \mathrm{Bar}_{\le 1} \to \cdots \to MO\langle 4n\rangle$$

factors through $X := \mathrm{colim}\,\mathrm{Bar}_2$ and fits well in the cube. $\square$

Summarised: from the diagram

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "1\otimes J"] \arrow[d, "\overset{(\text{product})}{m}"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J"] \arrow[dl, Leftrightarrow, "Ha"'] \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "J"] & \mathbb{S}
\end{tikzcd}$$

We obtain another diagram:

$$\begin{tikzcd}
\Sigma^\infty O\langle 4n-1\rangle \arrow[r, "1\otimes J - m"] & \Sigma^\infty O\langle 4n-1\rangle \arrow[r] \arrow[dr] & P \arrow[d, "\Omega"'] \\
 & & \mathbb{S} \arrow[d] \\
 & & C
\end{tikzcd}$$

cofib seq

cof seq.

S.t. $\tau_{\leq 12n-2}\, C \simeq \tau_{\leq 12n-2}\, MO\langle 4n\rangle$

&

$\tau_{\leq 12n-2}\,\mathbb{S} \to \tau_{\leq 12n-2}\, MO\langle 4n\rangle$ agrees with

$\tau_{\leq 12n-2}\,\mathbb{S} \longrightarrow \tau_{\leq 12n-2}\, C$

Consider the l.e.s of π_* of $P \to \mathbb{S} \to C$ and "replace" C by $MO\langle 4n\rangle$

$$\boxed{\pi_{8n-1}(P) \to \pi_{8n-1}(\mathbb{S}) \to \pi_{8n-1}(MO\langle 4n\rangle) \to \pi_{8n-2}(P) \to \pi_{8n-2}(\mathbb{S}).}$$

§ Homotopy of $\Sigma^\infty O\langle 4n-1\rangle$

Not Denote by $x \in \pi_{4n-1}(\Sigma^\infty O\langle 4n-1\rangle)$ a generator.

Since $\Sigma^\infty O\langle 4n-1\rangle$ is non-unital E_∞-ring, we have

$$x^2 \in \pi_{8n-2}(\Sigma^\infty O\langle 4n-1\rangle)$$

Also, denote by $J(x) \in \pi_{4n-1}(\mathbb{S})$ the composite

$$S^{4n-1} \xrightarrow{x} \Sigma^\infty O\langle 4n-1\rangle \xrightarrow{J} \mathbb{S}$$

Cor 4.8 i) $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle \cong \mathbb{Z}/2\mathbb{Z}\{x^2\}$

ii) $\pi_{8n-1}\Sigma^\infty O\langle 4n-1\rangle \cong \pi_{8n} ko \; (\cong \mathbb{Z})$

Lem 4.10 The element $xJ(x) \in \pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle$ is zero

Thm 4.11 For $4n-1 \leq l \leq 8n-1$, the image of

$$\pi_l(\Sigma^\infty O\langle 4n-1\rangle) \xrightarrow{J} \pi_l(\mathbb{S})$$

is $J_l = \mathrm{im}(\pi_l(O) \xrightarrow{J\text{-homo}} \pi_l(\mathbb{S}))$

Key ingredient for the proofs: Goodwillie calculus of the id of augmented E_∞-ring spectra.

Lemma 4.5 R E_∞-ring spectra with aug. $\varepsilon : R \to \mathbb{S}$ s.t. $\mathrm{fib}(\varepsilon)$ is 0-connected. The Goodwillie tower of id evaluated at R gives a convergent tower of E_∞-ring spectra:

$$\begin{tikzcd}
 & & D_n(TAQ(R;\mathbb{S})) \arrow[d] & & D_2(TAQ(R;\mathbb{S})) \arrow[d] & TAQ(R;\mathbb{S}) \arrow[d] & \mathbb{S} \arrow[d, "\simeq"] \\
R \arrow[r] & \cdots \arrow[r] & P_n(R) \arrow[r] & \cdots \arrow[r] & P_2(R) \arrow[r] & P_1(R) \arrow[r] & P_0(R)
\end{tikzcd}$$

s.t. $R \to P_0(R)$ is the augmentation ε. Here

$$D_n(TAQ(R,\mathbb{S})) := \left(TAQ(R,\mathbb{S})^{\otimes n}\right)_{h\Sigma_n}$$

Cor 4.6 We have convergent tower of spectra

$$\begin{tikzcd}
 & & D_n(\Sigma^{-1}\tau_{\geq 4n}ko) \arrow[d] & & D_3(\cdots) \arrow[d] & D_2(\cdots) \arrow[d] & \Sigma^{-1}\tau_{\geq 4n}ko \arrow[d, "\simeq"] \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r] & \cdots \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_3 \arrow[r] & Q_2 \arrow[r] & Q_1
\end{tikzcd} \qquad (*)$$

Pf Take $R = \Sigma^\infty_+ O\langle 4n-1\rangle$ with augmentation ε in the diagram in Lem 4.5

Fact: $TAQ(R,\mathbb{S}) \simeq \Sigma^{-1}\tau_{\geq 4n}ko$

Note that $\Sigma^\infty O\langle 4n-1\rangle \simeq \mathrm{fib}(\varepsilon)$ and set

$Q_n := \mathrm{fib}\,(P_n(R) \to P_{n-1}(R))$

Lem 4.7 – $\pi_{8n-2} D_2(\Sigma^{-1}\tau_{\geq 4n} ko) \cong \mathbb{Z}/2\mathbb{Z}$

$$\pi_{8n-1} D_2(\Sigma^{-1}\tau_{\geq 4n} ko) \cong 0$$

- The generator of $\pi_{8n-2} D_2(\Sigma^{-1}\tau_{\geq 4n} ko)$ survives in the SS associated to (*) and detects $x^2 \in \pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle$.

Cor 4.8 i) $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle \cong \mathbb{Z}/2\mathbb{Z}\{x^2\}$

ii) $\pi_{8n-1}\Sigma^\infty O\langle 4n-1\rangle \cong \pi_{8n} ko$ $(\cong \mathbb{Z})$

Sketch – $\tau_{\leq 8n-1} D_k(\Sigma^{-1}\tau_{\geq 4n} ko) \simeq *$ for $k>2$

$\rightsquigarrow$ a l.e.s. on π_* from (*):

$$\begin{tikzcd}
 & & & \mathbb{Z}/2\mathbb{Z} \arrow[d, phantom, "\cong" rotate=90] \\
0 \arrow[r] & \pi_{8n-1}(\Sigma^\infty O\langle 4n-1\rangle) \arrow[r] & \pi_{8n-1}(\Sigma^{-1}\tau_{\geq 4n} ko) \arrow[r] & \pi_{8n-2}(D_2(\Sigma^{-1}\tau_{\geq 4n} ko)) \arrow[d] \\
 & & 0 \arrow[d, phantom, "\cong" rotate=90] & \\
 & & \pi_{8n-2}(\Sigma^{-1}\tau_{\geq 4n} ko) & \pi_{8n-2}(\Sigma^\infty O\langle 4n-1\rangle) \arrow[l]
\end{tikzcd}$$

Claim $\pi_j(\Sigma^\infty O\langle 4n-1\rangle) \longrightarrow \pi_j(\Sigma^{-1}\tau_{\geq 4n} ko)$ are surjective

Sketch $\Sigma^\infty O\langle 4n-1\rangle \to \Sigma^{-1}\tau_{\geq 4n} ko$ is adjoint to

$$\mathrm{id}: O\langle 4n-1\rangle \to \Omega^{\infty+1}\tau_{\geq 4n} ko$$

$$\rightsquigarrow \quad \underbrace{\Omega^{\infty+1}\tau_{\geq 4n} ko \longrightarrow \Omega^\infty\Sigma^\infty O\langle 4n-1\rangle \overset{\curvearrowleft}{\longrightarrow} \Omega^{\infty+1}\tau_{\geq 4n} ko}_{\simeq \mathrm{id}\,\Sigma^{-1}\tau_{\geq 4n} ko}$$

$\square_{\text{claim}}$

$\rightsquigarrow$ Cor. follows by examining the l.e.s.

Lem 4.10 The element $xJ(x) \in \pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle$ is zero

Sketch Assume $xJ(x) \neq 0$, then $xJ(x) = x^2$, under the identification $\pi_{8n-2}\Sigma^\infty O\langle 4n-1\rangle \cong \mathbb{Z}/2\mathbb{Z}\{x^2\}$

Rmk 10.22: x^2 has $H\mathbb{F}_2$-adams filtration 1.

Recall that x is a generator of $\pi_{4n-1}\Sigma^\infty O\langle 4n-1\rangle$, and thus the suspension of an unstable class in $\pi_{4n-1} O\langle 4n-1\rangle$.

So $J(x)$ is in the image $\mathcal{J}_{4n-1}$ of J.

$\rightsquigarrow$ As $n > 3$, $J(x)$ has $H\mathbb{F}_2$-Adams filtration 3 and $xJ(x)$ has at least $H\mathbb{F}_2$-Adams filtration of $J(x)$ ↯ □

Thm 4.11 For $4n-1 \leq l \leq 8n-1$, the image of

$$\pi_l(\Sigma^\infty O\langle 4n-1\rangle) \xrightarrow{J} \pi_l(\mathbb{S})$$

is $\mathcal{J}_l = \mathrm{im}\left(\pi_l(O) \xrightarrow{J\text{-homo}} \pi_l(\mathbb{S})\right)$

Sketch It suffices to show that in this range, every $y \in \pi_l(\Sigma^\infty O\langle 4n-1\rangle)$ is the suspension of an unstable class.

Take a look at the Goodwillie tower (*) for $\Sigma^\infty O\langle 4n-1\rangle$

$$\begin{tikzcd}
 & & D_n(\Sigma^{-1}\tau_{\geq 4n}ko) \arrow[d] & & D_3(\cdots) \arrow[d] & D_2(\cdots) \arrow[d] & \Sigma^{-1}\tau_{\geq 4n}ko \arrow[d, "\cong"] \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r] & \cdots \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_3 \arrow[r] & Q_2 \arrow[r] & Q_1
\end{tikzcd} \qquad (*)$$

- for $4n-1 \leq l \leq 8n-3$, $\pi_l(D_k(\Sigma^{-1}\tau_{\geq 4n}ko)) \cong 0$, $k \geq 2$

  $\rightsquigarrow \pi_l(\Sigma^\infty O\langle 4n-1\rangle) \cong \pi_l(\Sigma^{-1}\tau_{\geq 4n}ko)$ ✓

- for $l = 8n-1$, <u>Cor 4.8</u> $\pi_{8n-1}(\Sigma^\infty O\langle 4n-1\rangle) \cong \pi_{8n}ko$ ✓

- for $l = 8n-2$, $\pi_{8n-2}(\Sigma^\infty O\langle 4n-1\rangle) \cong \mathbb{Z}/2\{x^2\}$

  <u>Prop. 15.11</u> $J(x^2) = J(x)^2 = 0$

$\square$