Prerequisites on sheaves of spectra

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<u>Goal of the paper</u> E- Adam type homology theory, e.g. E Landweber exact.

$\rightsquigarrow$ E_*E natural Hopf algebroid str.; E_*X is a comodule of E_*E.

Want to construct an derived ∞-cat of spectra, where the derivation is performed wrt the E-homology.

Object of this ∞-category: synthetic spectra (based on E): exhibits both the aspects of stable homotopy theory and homological algebra of comodules.

Def A synthetic spectra based on E is a spherical sheaf of spectra on the ∞-category Sp_E^{fp} of finite E_*-projective spectra

finite, f.g. proj. E_* modules

<u>Goal today</u>: A lot of prerequisites on spherical sheaves on additive ∞-sites.

§A.1. 2.1. 2.2. 2.3 of the main reference.

§ Grothendieck pretopology, sheaves etc.

<u>Def A.1</u> $\mathcal{C}$ ∞-cat. A Grothendieck pretopology on $\mathcal{C}$ is an assignment

$\tau: \mathrm{obj}(\mathcal{C}) \to \mathrm{Set}$ s.t. every element of $\tau(c)$ is a subset

$\{c_i \to c\}$ of $\mathrm{obj}(\mathcal{C}_{/c})$, called covers of c, s.t.

i) If $d \to c$ is an equivalence, then $\{d \to c\}$ is a cover of c

ii) If $\{c_i \to c\}$ is a cover and $d \to c \in \mathrm{Mor}(\mathcal{C})$, then

$\exists$ pullback $d \times_c c_i$, and $\{d \times_c c_i \to c\}$ is a cover of c

iii) If $\{c_i \to c\}$ and $\{c_{ij} \to c_i\}$ are covers, the the collection of composites $\{c_{ij} \to c_i \to c\}$ is a cover of c

Def. An ∞-site is an ∞-category equipped with Grothendieck pretopology.

Def A.10 $\mathcal{C}, \mathcal{D}$ ∞-sites. A morphism of ∞-site $f : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-cats s.t.

i) f preserves pullbacks along coverings, and

ii) If $\{c_i \to c\}$ is a covering of $c \in \mathrm{Obj}(\mathcal{C})$, then $\{f(c_i) \to f(c)\}$ is a covering in $\mathcal{D}$.

Def (Cor A.9) $\mathcal{C}$ ∞-site. $\mathcal{D}$ ∞-cat with finite products. Then a presheaf $X : \mathcal{C}^{op} \to \mathcal{D}$ is a sheaf iff $\forall$ covers $\{c_i \to c\} \in \tau(c)$, we have

$$X(c) \simeq \varprojlim \Big(\underbrace{\underbrace{\prod_i X(c_i) \rightrightarrows \prod_{i,j} X(c_i \times_c c_j)}_{\text{1-categorical sheaf condition}} \substack{\to\\\to\\\to} \cdots}_{\text{2-categorical sheaf condition} \cdots} \Big)$$

Rmk: $\mathrm{Pre}(\mathcal{C})$: ∞-cat of presheaves on $\mathcal{C}$ with values in spaces

$\mathcal{C} \hookrightarrow \mathrm{Pre}(\mathcal{C})$, $c \longmapsto \mathrm{Map}(-, c)$

$S \subseteq \mathrm{Mor}(\mathrm{Pre}(\mathcal{C}))$ consists of morphisms of the following form:

$$\mathrm{colim}\Big(\cdots \substack{\to\\\to\\\to} \coprod_{i,j} c_i \times_c c_j \rightrightarrows \coprod_i c_i \Big) \longrightarrow c, \quad \forall \{c_i \to c\} \in \tau(c)$$

$\mathrm{Sh}(\mathcal{C}) = S^{-1}(\mathrm{Pre}(\mathcal{C}))$, the ∞-cat. of S-local objects in $\mathrm{Pre}(\mathcal{C})$

$L : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}) = S^{-1}\mathrm{Pre}(\mathcal{C})$ sheafification, left adjoint to the forgetful functor.

Q: For which ∞-cat $\mathcal{D}$ is this true for $\mathrm{Sh}^{\mathcal{D}}(\mathcal{C})$?
cf. SAG 1.3.1.7

Prop A.11 Let $f: \mathcal{C} \to \mathcal{D}$ morphism of ∞-sites. The induced map $f^*: \mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{D})$ induces an adjunction

$$f_*: \mathrm{Sh}(\mathcal{C}) \rightleftarrows \mathrm{Sh}(\mathcal{D}) : f^*$$

where $f_* := L \circ \mathrm{Lan}$.

$\mathrm{Pre}(\mathcal{C})$, $\mathrm{Sh}(\mathcal{C})$
Presheaves / Sheaves of spaces

§ Spherical sheaves on additive ∞-categories

$\exists$ finite products and finite coproducts, and $h\mathcal{C}$ is additive

Def $\mathcal{C}$ ∞-cat with finite sums. $\mathcal{D}$ ∞-cat with finite products.
A presheaf $X: \mathcal{C}^{op} \to \mathcal{D}$ is **spherical** if it takes sums to products.

Not $\mathrm{Pre}^{\mathcal{D}}_{\Sigma}(\mathcal{C})$ $\mathrm{Sh}^{\mathcal{D}}_{\Sigma}(\mathcal{C})$

Lemma 2.1 $\mathcal{C}$ additive ∞-cat. Then $\mathrm{Pre}_{\Sigma}(\mathcal{C})$ is also additive.
In part. for any $X \in \mathrm{Pre}_{\Sigma}(\mathcal{C})$, X factors through $\mathrm{Alg}^{gr}_{E_\infty}(\mathcal{C})$

Cor 2.10

Ref Gepner–Groth–Nikolaus, Universality and Multiplicative of infinite loop space machine

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "X"] \arrow[d] & \mathcal{S}_* \\ \mathrm{Alg}^{gr}_{E_\infty}(\mathcal{S}_*) \arrow[ur] & \end{tikzcd}$$

infinite loop spaces

Def 2.3 An **additive ∞-site** is a small ∞-site and an additive ∞-cat where every cover consists of a single map, i.e. $\#\tau(X) = 1$ for every object X.

A **morphism** $f: \mathcal{C} \to \mathcal{D}$ **of additive ∞-sites** is a morphism of ∞-sites and a morphism of additive ∞-cats simultaneously

Prop 2.5 $\mathcal{C}$ be additive ∞-site. The sheafification $L: \mathrm{Pre}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ functors take spherical presheaves to spherical sheaves. This statement also holds for hypercomplete sheafification.

Pf Recall that L is the localization functor

$$L: \mathrm{Pre}(\mathcal{C}) \longrightarrow S^{-1}\mathrm{Pre}(\mathcal{C})$$

where S consists of morphisms:

single covering family

$$\varinjlim \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} d \times_c d \rightrightarrows d \right) \to c, \ \forall \{d \to c\} \in \tau(c)$$

+ presentable presheave are spherical
+ geometric realisation of spherical sheaves ~~are~~ is spherical

$\rightsquigarrow$ $S \subseteq \mathrm{Mor}(\mathrm{Pre}_\Sigma(\mathcal{C}))$

+ L preserves finite product (L is left exact)

$\rightsquigarrow$ L takes spherical presheaves to S-local spherical sheaves. Thm 2.2

$\square$

Cor 2.6 $\mathcal{C}$ additive ∞-site. $L_\Sigma: \mathrm{Pre}_\Sigma(\mathcal{C}) \to \mathrm{Sh}_\Sigma(\mathcal{C})$ presents $\mathrm{Sh}_\Sigma(\mathcal{C})$ as an accessible, left exact localization of $\mathrm{Pre}_\Sigma(\mathcal{C})$. In particular, $\mathrm{Sh}_\Sigma(\mathcal{C})$ is presentable. The statem also holds for hypercomplete sheaves.

presentable $\Rightarrow$ accessible

Sketch: Accessible: $\mathrm{Sh}_\Sigma(\mathcal{C}) = \mathrm{Sh}(\mathcal{C}) \cap \mathrm{Pre}_\Sigma(\mathcal{C})$ + HTT 5.4.7.10 + 5.5.1.2

L left exact $\rightsquigarrow$ L_Σ left exact

<u>Thm 2.8</u> (Recognition thm of spherical sheaves) $\mathcal{C}$ additive ∞-site: A spherical presheaf X is a (spherical) sheaf iff the following holds:

(*) If $F \to B \to A$ fib seq in $\mathcal{C}$ with $B \to A$ covering, then
$$X(A) \to X(B) \to X(F) \text{ is a fibre sequence in } \mathcal{S}_*$$

<u>Sketch</u>: – Assume X satisfies (*), want to show
$$X(A) \to X(B) \rightrightarrows X(B \times_A B) \substack{\to\\ \to\\ \to} \cdots \text{ is a limit diagram.}$$

Consider
$$\begin{tikzcd}
F := \mathrm{hofib} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{id}"'] \\
0 \arrow[r] & A \arrow[r] & A
\end{tikzcd}$$

coverings

fib seq

n-th terms

$$\begin{tikzcd}[column sep=small]
X(0) \arrow[r] & X(F) \arrow[r, shift left] \arrow[r, shift right] & X(F \times_0 F) \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots & X(F \times_0 \cdots \times_0 F) \arrow[r, shift left] \arrow[r, shift right] & \cdots \\
X(A) \arrow[u] \arrow[r] & X(B \times_A B) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & X(B \times_A B \times_A B) \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots & X(B \times_A \cdots \times_A B) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & \cdots \\
X(A) \arrow[u] \arrow[r] & X(A \times_A A) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & X(A \times_A A \times_A A) \arrow[u] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & \cdots & X(A \times_A \cdots \times_A A) \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & \cdots
\end{tikzcd}$$

fibre sequence of spaces

$$\rightsquigarrow \begin{tikzcd}
X(0) \arrow[r] \arrow[d, "\simeq"'] & X(A) \arrow[r] \arrow[d, "! \simeq"] & X(A) \arrow[d, "\simeq"] \\
\lim\limits_{\leftarrow} X(F \times_0 \cdots \times_0 F) \arrow[r] & \lim\limits_{\leftarrow} X(B \times_A \cdots \times_A B) \arrow[r] & \lim\limits_{\leftarrow} X(A \times_A \cdots \times_A A)
\end{tikzcd}$$

fib seq

$\sim$

uses: X take values in connective spectra (Lemma 2.1)

X is spherical presheaves:
$$X(F) \rightrightarrows X(F) \oplus X(F) \substack{\to\\ \to\\ \to} \cdots$$
cosimplicial object with extra degeneracies.

Compute limit of this cosimplicial space via BKSS $\rightsquigarrow$ contractible.

Converse $F \to B \to A$ fibre seq in $\mathcal{C}$, X spherical sheaf

Want $X(A) \to X(B) \to X(F)$ fib seq in $\mathcal{S}_*$

Consider

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\text{covers}"'] & B \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\
F \arrow[r] & B \arrow[r] & A
\end{tikzcd}$$

fib seq

$$\begin{tikzcd}
X(F) \arrow[r] & X(F) \arrow[r, shift left] \arrow[r, shift right] & X(F \times_F F) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \\
X(B) \arrow[r] \arrow[u] & X(B \times_A B) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & X((B \times_A B) \times_B (B \times_A B)) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u] & \cdots \\
X(A) \arrow[r] \arrow[u] & X(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & X(B \times_A B) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u] & \cdots
\end{tikzcd}$$

limit diagram

Enough to show each column is a fib seq.

$$\begin{tikzcd}
F \times_F \cdots \times_F F \arrow[r] & (B \times_A B) \times_B \cdots \times_B (B \times_A B) \arrow[r] & B \times_A B \times_A \cdots \times_A B \arrow[l, bend left, dashed]
\end{tikzcd}$$

$\Delta \times \cdots \times \Delta$ with $\Delta : B \to B \times_A B$

$\rightsquigarrow$ this is split exact fib seq

X spherical $\rightsquigarrow$ X preserves split exact sequences $\square$

Cor 2.9 $\mathcal{C}$ additive ∞-site. $Sh_\Sigma(\mathcal{C})$ is closed under filtered colimit. In particular filtered colimit of spherical sheaves are computed level wise. $\square$

Exercises

Some elementary results ~~on~~ of functoriality of $Sh_\Sigma(\mathcal{C})$

Prop 2.10 $f: \mathcal{C} \to \mathcal{D}$ morphism of additive ∞-sites. Then the induced adjunction $f_*: Sh(\mathcal{C}) \rightleftarrows Sh(\mathcal{D})$: f^* restricts to one on spherical sheaves. Analogue result holds in hypercomplete case.

Hint: use HTT 5.5.8.14

preserves small colimits.

Prop 2.11 $f^*: Sh_\Sigma(\mathcal{D}) \to Sh_\Sigma(\mathcal{C})$ is cocontinuous. Analogue holds in hypercomplete case

Hint: use Prop 2.9

Prop 2.12 f^* preserves n-connective and n-truncated obj's. (for later)

§ Sheaves of spectra

Prop 2.13 The ∞-cat $Sh^{Sp}_{\Sigma}(\mathcal{C})$ is the stabilisation of $Sh_{\Sigma}(\mathcal{C})$, thus (HA 1.4.4.4) a presentable stable ∞-cat.

Analogue holds in hypercomplete case.

Sketch. Fact: $Sh^{Sp}(\mathcal{C})$ is stabilisation of $Sh(\mathcal{C})$ (SAG. 1.3.2.2)

Thus

$$Sh^{Sp}(\mathcal{C}) \simeq \varprojlim(\cdots \to Sh(\mathcal{C})_* \xrightarrow{\Omega} Sh(\mathcal{C})_*)$$

$X \in Sh^{Sp}_{\Sigma}(\mathcal{C})$ iff $\Omega^{\infty-n}X \in Sh_{\Sigma}(\mathcal{C})$, since being spherical can be checked on finite limits.

$$\rightsquigarrow Sh^{Sp}_{\Sigma}(\mathcal{C}) \simeq \varprojlim(\cdots Sh_{\Sigma}(\mathcal{C}) \xrightarrow{\Omega} Sh_{\Sigma}(\mathcal{C}))$$

□

Def 2.15 - $X \in Sh^{Sp}_{\Sigma}(\mathcal{C})$. The n-th homotopy group $\pi_n X$ of X is defined as

$$\pi_n X := \text{sheafification} \circ (c \longmapsto \pi_n X(c)) \in Sh^{\text{disc.ab}}(X)$$

- X is connective if $\pi_n X = 0$ for $n < 0$ $\quad Sh^{Sp}_{\Sigma}(\mathcal{C})_{\geq 0}$
- X is coconnective if $\Omega^{\infty}X$ is a discrete sheaf of spaces $\quad Sh^{Sp}_{\Sigma}(\mathcal{C})_{\leq 0}$

Rmk. $\pi_n X$ is spherical for all n,

Prop 2.16 - $(Sh^{Sp}_{\Sigma}(\mathcal{C})_{\geq 0}, Sh^{Sp}_{\Sigma}(\mathcal{C})_{\leq 0})$ determines a right complete t-structure on $Sh^{Sp}_{\Sigma}(\mathcal{C})$ Ref. 1.3.2 in SAG for generalisation

- this t-structure is compatible with filtered colimits: $Sh^{Sp}_{\Sigma}(\mathcal{C})_{\leq 0}$ is closed under filtered colimit.

- ∃ canonical equivalence $Sh^{Sp}_{\Sigma}(\mathcal{C})^{\heartsuit} \simeq Sh^{Set}_{\Sigma}(\mathcal{C}) \underset{\text{additivity of } \mathcal{C}}{\simeq} Sh^{Ab}_{\Sigma}(\mathcal{C})$

Sketchy sketch: + The same statement holds for $Sh^{Sp}(\mathcal{C})$, SAG 1.3.2.7

+ $(-)_{\geq 0}$ and $(-)_{\leq 0}$ preserve spherical sheaves

+ right completeness: (?)

$$\operatorname{colim}\left(\cdots \to (Sh_\Sigma^{Sp}\mathcal{C})_{\geq 1} \to (Sh_\Sigma^{Sp}\mathcal{C})_{\geq 0} \to \cdots\right) \simeq Sh_\Sigma^{Sp}(\mathcal{C})$$

check some formal properties

$\square$

Prop 2.19 Consider the adjunction

computed levelwise

$$\Sigma^\infty_+ : Sh_\Sigma(\mathcal{C}) \rightleftarrows Sh_\Sigma^{Sp}(\mathcal{C}) : \Omega^\infty .$$

The functor Σ^∞_+ is fully faithful and

$$\Sigma^\infty_+(Sh_\Sigma(\mathcal{C})) \simeq Sh_\Sigma^{Sp}(\mathcal{C})_{\geq 0}$$

Analogue holds for hyper completion

Sketch: By Lemma 2.1: $\Omega^\infty : Pre_\Sigma^{Sp_{\geq 0}}(\mathcal{C}) \xrightarrow{\simeq} Pre_\Sigma(\mathcal{C})$

$$\rightsquigarrow Sh_\Sigma^{Sp_{\geq 0}}(\mathcal{C}) \simeq Sh_\Sigma(\mathcal{C})$$

By SAG 1.3.5.7 $(-)_{\geq 0} : Sp \to Sp_{\geq 0}$ induces adjoint equiv

$$Sh^{Sp}(\mathcal{C})_{\geq 0} \simeq Sh^{Sp_{\geq 0}}(\mathcal{C})$$

+ $(-)_{\geq 0}$ preserves spherical sheaves

$\square$

Prop 2.21 $f : \mathcal{C} \to \mathcal{D}$ morphism of additive ∞-sites.

The adjunction $Sh_\Sigma(\mathcal{C}) \rightleftarrows Sh_\Sigma(\mathcal{D})$ induces an adj

$$f_* : Sh_\Sigma^{Sp}(\mathcal{C}) \rightleftarrows Sh_\Sigma^{Sp}(\mathcal{D}) : f^*$$

s.t. $f_*(\Sigma^\infty_+ c) \simeq \Sigma^\infty_+(f(c))$ and f^* is cocontinuous

Rmk 2.22 f^* preserves both connective and coconnective part of the t-structure.

§ Sym monoidal Structure on $Sh_\Sigma(\mathcal{C})$ and $Sh_\Sigma^{Sp}(\mathcal{C})$

Prop 2.24 $\mathcal{C}$ sym mon ∞-cat. Then $Pre(\mathcal{C})$ admits a unique sym. mon. str. s.t. $\mathcal{C} \hookrightarrow Pre(\mathcal{C})$ is sym monoidal and $\otimes : Pre(\mathcal{C}) \otimes Pre(\mathcal{C}) \to Pre(\mathcal{C})$ preserves colimits in each variable.

Pf. H.A. 4.8.1 using Day Convolution

Q: When does Day convolution compatible with the localization

$$Pre(\mathcal{C}) \longrightarrow Sh_\Sigma(\mathcal{C}) \quad ?$$

Def 2.23 An excellent ∞-site $\mathcal{C}$ is an additive ∞-site equipped with a sym mon. str s.t. all objs admits dual and $\forall c \in Ob(\mathcal{C})$, the functor

$$- \otimes c : \mathcal{C} \to \mathcal{C} \quad \text{takes covers to covers}$$

From now on $\mathcal{C}$ excellent ∞-site

Thm 2.27 The Day Convolution sym mon str. on $Pre(\mathcal{C})$ preserves L_X-equivalence in both variables where $L_X : Pre(\mathcal{C}) \to X$ with $X \in \{ Sh^{Set}(\mathcal{C}), Sh_\Sigma^{Set}(\mathcal{C}), Sh(\mathcal{C}), Sh_\Sigma(\mathcal{C}), \widehat{Sh}(\mathcal{C}), \widehat{Sh}_\Sigma(\mathcal{C}) \}$ is the left adjoint of the inclusion $X \hookrightarrow Pre(\mathcal{C})$

Cor 2.28 All of X above admits a unique sym mon str. which preserves colimit in each variable, and

$$L_X \circ i : \mathcal{C} \xrightarrow{i} \mathrm{Pre}(\mathcal{C}) \xrightarrow{L_X} X$$

is sym. monoidal

Pf H.A. 2.2.1.9.

Prop 2.29 $Sh^{Sp}_\Sigma(\mathcal{C})$ admits a sym monoidal structure which is cocontinuous in each variable and

$$\Sigma^\infty_+ : Sh_\Sigma(\mathcal{C}) \longrightarrow Sh^{Sp}_\Sigma(\mathcal{C})$$

is sym. monoidal. Analogue result hold in hypercomplete case

Sketch: Pr^L : ∞-cat of presentable ∞-cats and cocontinuous functors

Fact: $Alg_{E_\infty}(Pr^L)$ = sym monoidal presentable ∞-cats s.t. $\otimes$ is cocontinuous in each variable

$\rightsquigarrow$ $Sh_\Sigma(\mathcal{C}) \in Alg_{E_\infty}(Pr^L)$ and $Sp \in Alg_{E_\infty}(Pr^L)$ $\rightsquigarrow$ $Sh_\Sigma(\mathcal{C}) \otimes Sp \in Alg_{E_\infty}(Pr^L)$

$\xrightarrow{\text{H.A. 4.8.2.18}}$ $Sh_\Sigma(\mathcal{C}) \otimes Sp$ is stabilization of $Sh_\Sigma(\mathcal{C})$ $\square$