

Prerequisites on sheaves of spectra

30. Nov. 2020. Yuqing Shi

Goal of the paper E-Adams type homology theory, e.g. E Landweber exact

$\Rightarrow E_*E$ natural Hopf algebroid str.; E_*X is a comodule of E_*E .

Want to construct an derived ∞ -cat of spectra, where the derivation is performed w.r.t the E -homology.

object of this ∞ -category: synthetic spectra (based on E): exhibits both the aspects of stable homotopy theory and homological algebra of comodules

Def A **Synthetic Spectra** based on E is a Spherical sheaf of Spectra on the ∞ -category $S_{E, \text{proj}}^{\text{fin}}$ of finite E_* -projective spectra finite, f.g. proj. E_* modules

Goal today: A lot of prerequisites on spherical sheaves on additive ∞ -sites.

§A.1. 2.1. 2.2. 2.3 of the main reference.

§ Grothendieck pretopology, sheaves, etc.

Def A.1 \mathcal{C} ∞ -cat. A **Grothendieck pretopology** on \mathcal{C} is an assignment

$\tau: \text{obj}(\mathcal{C}) \rightarrow \text{Set}$ s.t. every element of $\tau(c)$ is a subset

$\{c_i \rightarrow c\}$ of $\text{obj}(\mathcal{C}/c)$, called **covers** of c . s.t.

i) If $d \rightarrow c$ is an equivalence, then $\{d \rightarrow c\}$ is a cover of c

ii) If $\{c_i \rightarrow c\}$ is a cover and $d \rightarrow c \in \text{Mor}(\mathcal{C})$, then \exists pullback $d \times_c c_i$, and $\{d \times_c c_i \rightarrow c\}$ is a cover of c

iii) If $\{c_i \rightarrow c\}$ and $\{c_{ij} \rightarrow c_i\}$ are covers, then the collection of composites $\{c_{ij} \rightarrow c_i \rightarrow c\}$ is a cover of c

Def. An **∞ -site** is an ∞ -category equipped with Grothendieck pretopology.

Def A.6 \mathcal{C}, \mathcal{D} ∞ -sites. A **morphism of ∞ -site** $f: \mathcal{C} \rightarrow \mathcal{D}$ is a functor of ∞ -cats s.t.

- i) f preserves pullbacks along coverings, and
- ii) If $\{c_i \rightarrow c\}$ is a covering of $c \in \text{obj}(\mathcal{C})$, then $\{f(c_i) \rightarrow f(c)\}$ is a covering in \mathcal{D} .

Def (Cor A.9) \mathcal{C} ∞ -site. \mathcal{D} ∞ -cat with finite products. Then a presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ is a **sheaf** iff \forall covers $\{c_i \rightarrow c\} \in \tau(c)$, we have

$$X(c) \cong \lim_{\leftarrow} \left(\prod_i X(c_i) \rightrightarrows \prod_{i,j} X(c_i \times_c c_j) \rightrightarrows \dots \right)$$

1-categorical sheaf condition
2-categorical sheaf condition

Remark: $\text{Pre}(\mathcal{C})$: ∞ -cat of presheaves on \mathcal{C} with values in spaces
 $\mathcal{C} \hookrightarrow \text{Pre}(\mathcal{C}), c \mapsto \text{Map}(-, c)$

$S \subseteq \text{Mor}(\text{Pre}(\mathcal{C}))$ consists of morphisms of the following form:

$$\text{colim} \left(\dots \rightrightarrows \prod_{i,j} c_i \times_c c_j \rightrightarrows \prod_i c_i \right) \rightarrow c, \forall \{c_i \rightarrow c\} \in \tau(c)$$

$\text{Sh}(\mathcal{C}) = S^{-1}(\text{Pre}(\mathcal{C}))$, the ∞ -cat of **S -local** objects in $\text{Pre}(\mathcal{C})$

$L: \text{Pre}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{C}) = S^{-1}\text{Pre}(\mathcal{C})$ **sheafification**, left adjoint to the forgetful functor.

Q. For which ∞ -cat \mathcal{D} is this true for $\text{Sh}^{\mathcal{D}}(\mathcal{C})$?
 c.f. SAG 1.3.1.7.

Prop. A.11 Let $f: \mathcal{C} \rightarrow \mathcal{D}$ morphism of ∞ -sites. The induced map

$f^*: \text{Pre}(\mathcal{C}) \rightarrow \text{Pre}(\mathcal{D})$ induces an adjunction

$$f_*: \text{Sh}(\mathcal{C}) \rightleftarrows \text{Sh}(\mathcal{D}) : f^*$$

where $f_* := L \circ \text{Lan}$.

$\text{Pre}(\mathcal{C}), \text{Sh}(\mathcal{C})$
 Presheaves/Sheaves
 of spaces

\exists finite products and finite coproducts, and $h\mathcal{C}$ is additive

Spherical sheaves on additive ∞ -categories

Def \mathcal{C} ∞ -cat with finite sums. \mathcal{D} ∞ -cat with finite products.

A presheaf $X: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ is **spherical** if it takes sums to products.

Not $\text{Pre}_{\Sigma}^{\mathcal{D}}(\mathcal{C})$ $\text{Sh}_{\Sigma}^{\mathcal{D}}(\mathcal{C})$

Lemma 1 \mathcal{C} additive ∞ -cat. Then $\text{Pre}_{\Sigma}(\mathcal{C})$ is also additive.

In part. for any $X \in \text{Pre}_{\Sigma}(\mathcal{C})$, X factors through $\text{Alg}_{\mathbb{E}_{\infty}}^{\text{gr}}(\mathcal{C})$

Cor. 2.10

Ref Gepner-broch-Nikolaus, $\mathcal{C}^{\text{op}} \xrightarrow{X} \mathcal{S}_*$

Universality and Multiplicativity
 of infinite loop space machines

$\downarrow \text{Alg}_{\mathbb{E}_{\infty}}^{\text{gr}}(\mathcal{S}_*) \uparrow$ \leftarrow infinite loop spaces

Def 2.3 An **additive ∞ -site** is a small ∞ -site and an additive ∞ -cat. where every cover consists of a single map, i.e. $\# \mathcal{C}(X) = 1$ for every object X .

A **morphism $f: \mathcal{C} \rightarrow \mathcal{D}$ of additive ∞ -sites** is a morphism of ∞ -sites and a morphism of additive ∞ -cats simultaneously

Prop 2.5 \mathcal{C} be additive ∞ -site. The sheafification $L: \text{Pre}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{C})$ functors take spherical presheaves to spherical sheaves. This statement also holds for hypercomplete sheafification.

Pf Recall that L is the localisation functor

$$L: \text{Pre}(\mathcal{C}) \rightarrow S^{-1} \text{Pre}(\mathcal{C})$$

where S consists of morphisms:

$$\lim_{\rightarrow} (\dots \rightarrow d \times_c d \rightarrow d) \rightarrow c, \forall \{d \rightarrow c\} \in \mathcal{C}(c)$$

single covering family

+ presentable presheaves are spherical

+ geometric realisation of spherical sheaves ~~are~~ is spherical

$$\leadsto S \subseteq \text{Mor}(\text{Pre}_{\Sigma}(\mathcal{C}))$$

+ L preserves finite product (L is left exact)

$\hookrightarrow L$ takes spherical presheaves to S -local spherical sheaves. Thm 2.2 \square

Cor 2.6 \mathcal{C} additive ∞ -site. $L_{\Sigma}: \text{Pre}_{\Sigma}(\mathcal{C}) \rightarrow \text{Sh}_{\Sigma}(\mathcal{C})$ presents $\text{Sh}_{\Sigma}(\mathcal{C})$ as an accessible, left exact localisation of $\text{Pre}_{\Sigma}(\mathcal{C})$. In particular, $\text{Sh}_{\Sigma}(\mathcal{C})$ is presentable. The statement also holds for hypercomplete sheaves.

presentable \Rightarrow accessible

Sketch: Accessible: $\text{Sh}_{\Sigma}(\mathcal{C}) = \text{Sh}(\mathcal{C}) \cap \text{Pre}_{\Sigma}(\mathcal{C})$ + HTT 5.4.7.10 + 5.5.1.2

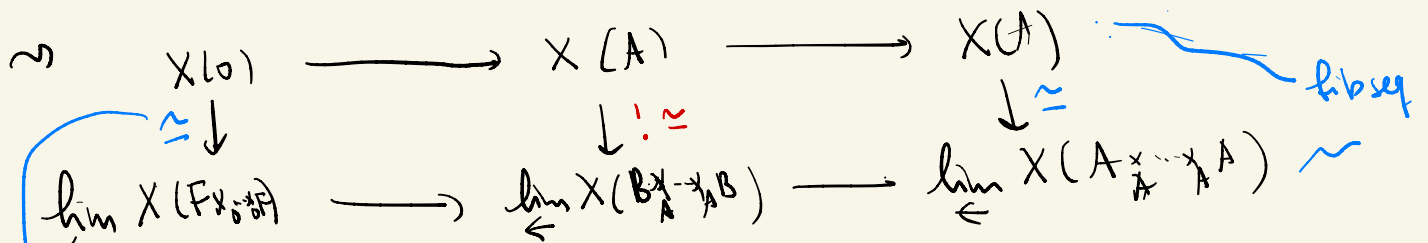
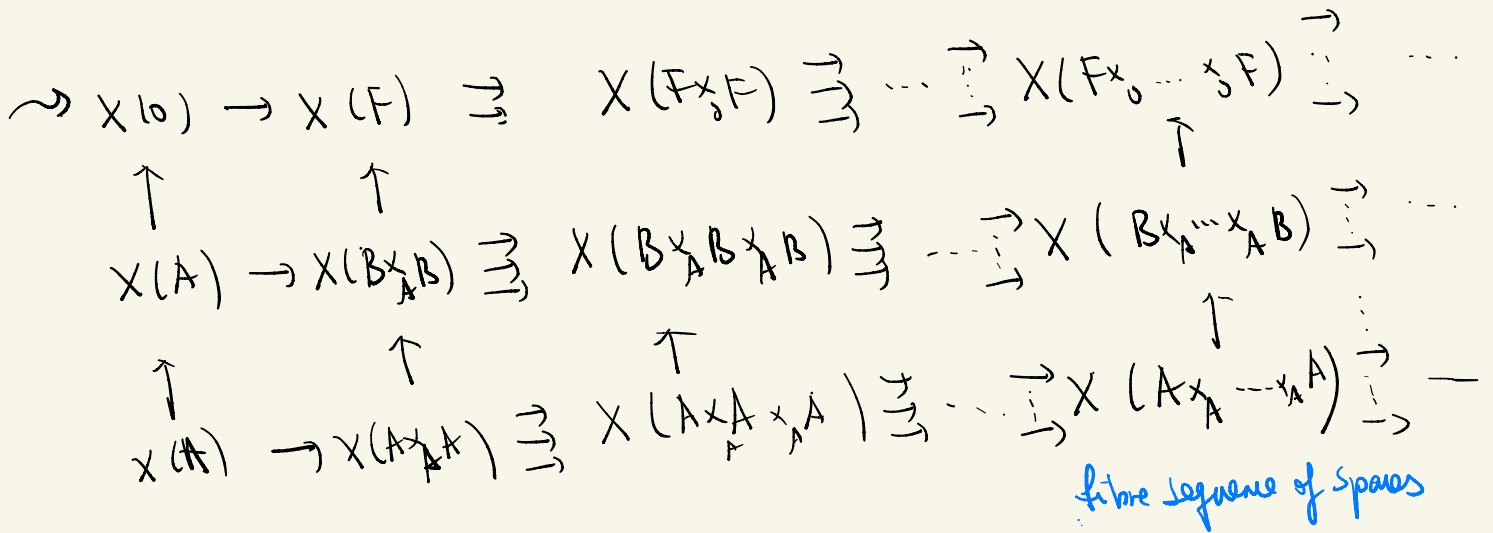
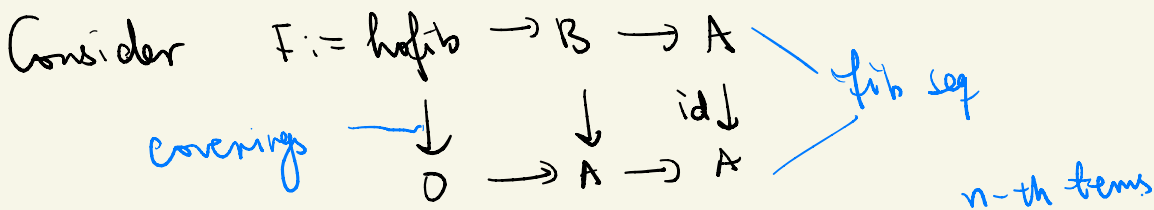
L left exact $\leadsto L_{\Sigma}$ left exact

Thm 2f (Recognition thm of spherical sheaves) \mathcal{C} additive co-site: A spherical presheaf X is a (spherical) sheaf iff the following holds:

(*) If $F \rightarrow B \rightarrow A$ fib seq in \mathcal{C} with $B \rightarrow A$ covering, then $X(A) \rightarrow X(B) \rightarrow X(F)$ is a fibre sequence in \mathcal{S}^*

Sketch: - Assume X satisfies (*), want to show

$X(A) \rightarrow X(B) \rightrightarrows X(B \times_A B) \rightrightarrows \dots$ is a limit diagram.



uses: X take values in connective spectra (Lem 2.1)

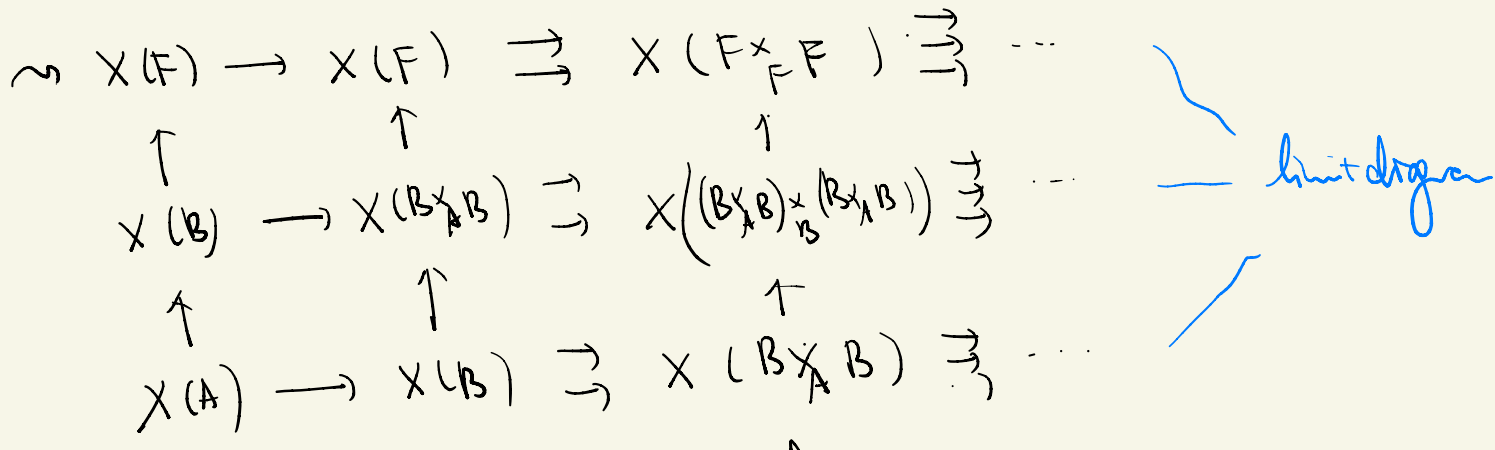
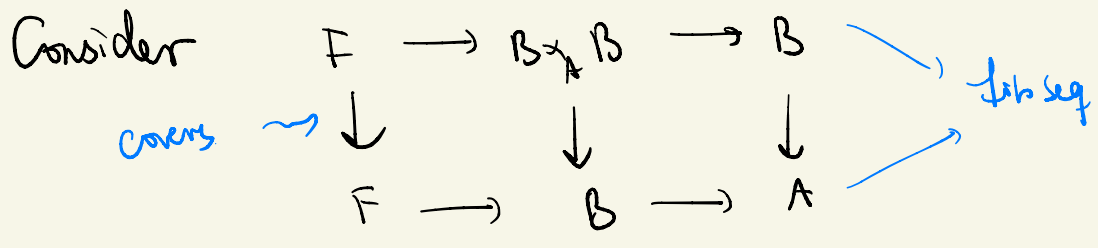
X is spherical presheaves:

$$X(F) \rightrightarrows X(F) \oplus X(F) \rightrightarrows \dots$$

cosimplicial object with extra degeneracies.

Compute limit of this cosimplicial space via BSS \leadsto contractible.

Converse $F \rightarrow B \rightarrow A$ fibre seq in \mathcal{C} , X spherical sheaf
 Want $X(A) \rightarrow X(B) \rightarrow X(F)$ fib seq in \mathcal{S}_*



Enough to show each column is a fib seq.

$$F \times_F \dots \times_F F \longrightarrow (B \times_A B) \times_B \dots \times_B (B \times_A B) \longrightarrow B \times_A B \times_A \dots \times_A B$$

$\Delta \times \dots \times \Delta$ with $\Delta: B \rightarrow B \times_A B$

\rightsquigarrow this is split exact fib seq

X spherical $\rightsquigarrow X$ preserves split exact sequences □

Cor 2.9 \mathcal{C} additive co-site. $Sh_{\Sigma}(\mathcal{C})$ is closed under filtered colimit. In particular ~~filtered~~ colimit of spherical sheaves are computed level wise. □

Exercises

Some elementary results on of functionality of $Sh_{\Sigma}(C)$

Prop 2.10 $f: C \rightarrow D$ morphism of additive ∞ -sites. Then the induced adjunction $f_*: Sh(C) \rightleftharpoons Sh(D)$: f^* restricts to one on spherical sheaves. Analogue result holds in hypercomplete case.

Hint: use HTT 5.5.8.14

preserves small colimits.



Prop 2.11

$f^*: Sh_{\Sigma}(D) \rightarrow Sh_{\Sigma}(C)$ is locally finitely presentable. Analogue holds in hypercomplete case

Hint: use Prop 2.19

Prop 2.12

f^* preserves n -connective and n -truncated obj's. (for later)

§ Sheaves of Spectra

Prop 2.13 The ∞ -cat $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})$ is the stabilisation of $\mathrm{Sh}_{\Sigma}(\mathcal{C})$, thus (HA.1.4.4.4) a presentable stable ∞ -cat. Analogue holds in hypercomplete case.

Sketch: Fact: $\mathrm{Sh}^{\mathrm{Sp}}(\mathcal{C})$ is stabilisation of $\mathrm{Sh}(\mathcal{C})$ (SAG. 1.3.2.2)

Thus

$$\mathrm{Sh}^{\mathrm{Sp}}(\mathcal{C}) \simeq \varprojlim (\dots \rightarrow \mathrm{Sh}(\mathcal{C})_* \xrightarrow{\Omega} \mathrm{Sh}(\mathcal{C})_*)$$

$X \in \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})$ iff $\Omega^{\infty-n} X \in \mathrm{Sh}_{\Sigma}(\mathcal{C})$, since being spherical can be checked on finite limits.

$$\Rightarrow \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C}) \simeq \varprojlim (\dots \mathrm{Sh}_{\Sigma}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Sh}_{\Sigma}(\mathcal{C}))$$

□

Def 2.15 - $X \in \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})$. The n -th homotopy group $\pi_n X$ of X is defined as

$$\pi_n X := \mathrm{Shaffixation}_0 (c \mapsto \pi_n X(c)) \in \mathrm{Sh}^{\mathrm{discrete}}(X)$$

- X is **connective** if $\pi_n X = 0$ for $n < 0$ $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})_{\geq 0}$
- X is **coconnective** if $\Omega^{\infty} X$ is a discrete sheaf of spaces $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})_{\leq 0}$

Remark. $\pi_n X$ is spherical for all n .

Prop 2.16 - $(\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})_{\geq 0}, \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})_{\leq 0})$ determines a right complete

t -structure on $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})$ Ref. 1.3.2 in SAG for generalisation

- this t -structure is compatible with filtered colimits: $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})_{\leq 0}$ is closed under filtered colimit.

- \exists canonical equivalence $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathcal{C})^{\heartsuit} \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Set}}(\mathcal{C}) \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Ab}}(\mathcal{C})$ additivity of \mathcal{C}

Sketchy sketch: + The same statement holds for $\text{Sh}^{\text{Sp}}(\mathcal{C})$, SAT 1.3.2.7

+ $(-)\gg_0$ and $(-)\leq_0$ preserve spherical sheaves.

+ right completeness: (?)

$$\text{colim}(\dots \rightarrow (\text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C}))_{\gg_1} \rightarrow (\text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C}))_{\gg_0} \rightarrow \dots) \simeq \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C})$$

check some formal properties

□

Prop 2.19 Consider the adjunction

$$\Sigma_+^{\infty} : \text{Sh}_{\Sigma}(\mathcal{C}) \rightleftarrows \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C}) : \Omega^{\infty}$$

← computed because

The functor Σ_+^{∞} is fully faithful and

$$\Sigma_+^{\infty}(\text{Sh}_{\Sigma}(\mathcal{C})) \simeq \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C})_{\gg_0}$$

Analogue holds for hyper completion

Sketch: by Lemma 2.1 : $\Omega^{\infty} : \text{Pre}_{\Sigma}^{\text{Sp}\gg_0}(\mathcal{C}) \xrightarrow{\simeq} \text{Pre}_{\Sigma}(\mathcal{C})$

$$\xrightarrow{\simeq} \text{Sh}_{\Sigma}^{\text{Sp}\gg_0}(\mathcal{C}) \simeq \text{Sh}_{\Sigma}(\mathcal{C})$$

By SAT 1.3.5.7 $(-)\gg_0 : \text{Sp} \rightarrow \text{Sp}\gg_0$ induces adjoint equiv

$$\text{Sh}^{\text{Sp}}(\mathcal{C})_{\gg_0} \simeq \text{Sh}^{\text{Sp}\gg_0}(\mathcal{C})$$

+ $(-)\gg_0$ preserves spherical sheaves

□

Prop 2.21 $f : \mathcal{C} \rightarrow \mathcal{D}$ morphism of additive ∞ -sites.

The adjunction $\text{Sh}_{\Sigma}(\mathcal{C}) \rightleftarrows \text{Sh}_{\Sigma}(\mathcal{D})$ induces an adj

$$f_* : \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C}) \rightleftarrows \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{D}) : f^*$$

st. $f_*(\Sigma_+^{\infty} c) \simeq \Sigma_+^{\infty}(f(c))$ and f^* is cocontinuous

Prop 2.22 f^* preserves both connective and coconnective part of the t -structure.

§ Sym monoidal Structure on $\mathrm{Sh}_{\Sigma}(\mathcal{C})$ and $\mathrm{Sh}_{\Sigma}^{\mathrm{sp}}(\mathcal{C})$

Prop 2.24 \mathcal{C} sym mon. ∞ -cat. Then $\mathrm{Pre}(\mathcal{C})$ admits a unique sym mon. str. s.t. $\mathcal{C} \hookrightarrow \mathrm{Pre}(\mathcal{C})$ is sym monoidal and $\otimes: \mathrm{Pre}(\mathcal{C}) \otimes \mathrm{Pre}(\mathcal{C}) \rightarrow \mathrm{Pre}(\mathcal{C})$ preserves colimits in each variable.

Pf. HA. 4.8.1 using Day Convolution

Q: When does Day Convolution compatible with the localisation $\mathrm{Pre}(\mathcal{C}) \rightarrow \mathrm{Sh}_{\Sigma}(\mathcal{C})$?

Def 2.23 An excellent ∞ -site \mathcal{C} is an additive ∞ -site equipped with a sym mon. str. s.t. all obj's admits dual and $\forall c \in \mathrm{Obj}(\mathcal{C})$, the functor $- \otimes c: \mathcal{C} \rightarrow \mathcal{C}$ takes covers to covers

From now on \mathcal{C} excellent ∞ -site

Thm 2.27 The Day Convolution sym mon. str. on $\mathrm{Pre}(\mathcal{C})$ preserves L_X -equivalence in both variables where $L_X: \mathrm{Pre}(\mathcal{C}) \rightarrow \mathcal{X}$ with $\mathcal{X} \in \{ \mathrm{Sh}^{\mathrm{set}}(\mathcal{C}), \mathrm{Sh}_{\Sigma}^{\mathrm{set}}(\mathcal{C}), \mathrm{Sh}(\mathcal{C}), \mathrm{Sh}_{\Sigma}(\mathcal{C}), \widehat{\mathrm{Sh}}(\mathcal{C}), \widehat{\mathrm{Sh}}_{\Sigma}(\mathcal{C}) \}$ is the left adjoint of the inclusion $\mathcal{X} \hookrightarrow \mathrm{Pre}(\mathcal{C})$

Cor 2.28 All of \mathcal{X} above admits a unique sym mon str. which preserves colimit in each variable, and

$$L_{\mathcal{X}} \circ i: \mathcal{C} \xrightarrow{i} \text{Pre}(\mathcal{C}) \xrightarrow{L_{\mathcal{X}}} \mathcal{X}$$

is sym. monoidal

PF H.A. 2.2.1.9.

Prop 2.29 $\text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C})$ admits a sym monoidal structure which is cocontinuous in each variable and

$$\Sigma_+^{\infty}: \text{Sh}_{\Sigma}(\mathcal{C}) \rightarrow \text{Sh}_{\Sigma}^{\text{Sp}}(\mathcal{C})$$

is sym. monoidal. Analogue result hold in hypercomplex

Sketch: Pr^L : ∞ -cat of presentable ∞ -cats and cocontinuous functors

Fact: $\text{Alg}_{\text{E}_{\infty}}(\text{Pr}^L) = \text{Sym monoidal presentable } \infty\text{-cats st. } \otimes$
is cocontinuous in each variable

$$\leadsto \text{Sh}_{\Sigma}(\mathcal{C}) \in \text{Alg}_{\text{E}_{\infty}}(\text{Pr}^L)$$

$$\text{and } S_p \in \text{Alg}_{\text{E}_{\infty}}(\text{Pr}^L)$$

$$\leadsto \text{Sh}_{\Sigma}(\mathcal{C}) \otimes S_p \in \text{Alg}_{\text{E}_{\infty}}(\text{Pr}^L)$$

H.A. 4.8.2.18 $\rightarrow \text{Sh}_{\Sigma}(\mathcal{C}) \otimes S_p$ is stabilization of $\text{Sh}_{\Sigma}(\mathcal{C})$

□