Main Results

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Notation:

p = a prime

E = a p-local Landweber exact homology theory of height n.

THM:

① Assume that $p > n^2+n+1$, then there exists an equivalence

$$\underbrace{hSp_E}_{\substack{\infty\text{-cat. of}\\ E\text{-local spectral}}} \simeq h\underbrace{D(E_*E)}_{\substack{\infty\text{-cat. of differential}\\ E_*E\text{-comodules (will specify later)}}}$$

② The statement becomes stronger the larger the prime:

$p > n^2+n+1+\frac{k}{2}$, where $k \geq 1$

$$h_k Sp_E \simeq h_k D(E_*E)$$

Recall: A <u>htpy k-category</u> is an ∞-category whose underlying mapping spaces are $(k-1)$-truncated.

$$h_k : \underset{(\infty,1)}{Cat_\infty} \longrightarrow \underset{(k,1)}{Cat_k} \quad \text{is a localization}$$

- The derived ∞-category $D(E_*E)$

$dComod_{E_*E}$ = (ordinary) category of differential E_*E-comodules.

Barnes - Roitzheim

$$dComod_{E_*E} \simeq Ch^{per}(Comod_{E_*E}) \simeq Mod_{P(\mathbb{1})}(Ch(Comod_{E_*E}))$$

Prop 3.3

$P(\mathbb{1}) = E_*[\tau^{\pm 1}]$

$\tau = (1,-1)$

Def 3.5

We define $D(E_*E)$ as the underlying symmetric monoidal ∞-category of $dComod_{E_*E}$, called the derived ∞-cat. of E_*E.

Warning: $D(E_*E) \not\simeq D(Comod_{E_*E})$

$$D(E_*E) \simeq Mod_{P(\mathbb{1})}(D(Comod_{E_*E}))$$

Rem:

- For $n=1$, Bousfield proved the statement for $E = KU_{(p)}$.

- Franke attempted to generalize this statement to all heights, but there was a subtle error found by Patchkoria.

- The equivalence

$$hSp_E \simeq hD(E_*E)$$

does not come from an equivalence $Sp_E \simeq D(E_*E)$.

THM: (Barthel – Schlank – Stapleton)

$$Sp_E \not\simeq D(E_*E) \text{ at any prime.}$$

- The number n^2+n comes from the following key observation:

Thm 2.4 :

For $p > n+1$, the abelian category Comod_{E_*E} has homological dimension n^2+n, that is

$$\mathrm{Ext}^{s,t}(X, Y) = 0 \qquad \text{for } s > n^2+n$$

for any $X, Y \in \mathrm{Comod}_{E_*E}$.

Overview of the Approach

Use Goerss-Hopkins obstruction theory, to construct:

$$\lim \mathcal{M}_e^{top} \simeq Sp_E$$

$$\begin{tikzcd}
\mathcal{M}_\infty^{top} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{M}_e^{top} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{M}_1^{top} \arrow[r] & \mathcal{M}_0^{top} \arrow[dd, "S"] \\
h_k\mathcal{M}_\infty^{top} \arrow[rr, "\simeq"] & & h_k\mathcal{M}_e^{top} \arrow[d, "S"'] & & & \\
h_k\mathcal{M}_\infty^{alg} \arrow[rr, "\simeq"] & & h_k\mathcal{M}_e^{alg} & & & D(\mathrm{Comod}_{E_*E}) \\
\mathcal{M}_\infty^{alg} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{M}_e^{alg} \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{M}_1^{alg} \arrow[r] & \mathcal{M}_0^{alg} \arrow[u, "S"']
\end{tikzcd}$$

Bousfield adjunction

$$\mathcal{M}_\infty^{alg} \simeq \lim \mathcal{M}_e^{alg} \simeq D(E_*E)$$

1. Construct the topological tower & the algebraic tower

2. Construct the Bousfield adjunction & related functors

3. : Finish the proof

Recollection from Synthetic spectra

Recall:

- The ∞-cat. of synthetic spectra

spectral valued

$$\mathrm{Syn} := \mathrm{Sh}^{Sp}_{\Sigma}(\underbrace{Sp_E^{t.p.}})$$

spherical $\qquad$ ∞-cat. of fin. proj. spectra based at E.

- There is a fully faithful embedding:

$$\nu : Sp \longrightarrow \mathrm{Syn}$$

- there is also a hypercomplete version of synthetic spectra

$$\widehat{\mathrm{Syn}} := \widehat{\mathrm{Sh}}^{Sp}_{\Sigma}(Sp_E^{t.p.})$$

Prop 5.4 [Pst 18b]

$\widehat{\mathrm{Syn}}$ is the localization of Syn at the class of

νE-equivalences.

Facts :

- $\widehat{Syn}$ inherits a right complete t-structure from Syn.
- $\widehat{Syn}^\heartsuit \simeq \mathrm{Comod}_{E_*E}$

$X \in Syn$

$\pi_*^\heartsuit X := (\nu E)_{*,*}(X)$

chow deg, top. deg, weight

- A map $f: X \to Y$ in $\widehat{Syn}$ is an equivalence

$\Leftrightarrow$ $\pi_*^\heartsuit(f) : \pi_*^\heartsuit X \to \pi_*^\heartsuit Y$ is an isom.

- $\widehat{Syn}$ inherits a sym. mon. structure from Syn.

in which the unit object is $\nu S_E =: \mathbb{1}$.

$$\pi_*^\heartsuit(\mathbb{1}) = E_*[\tau] \quad , |\tau| = (1,-1)$$

$(X \hat{\otimes} Y) \simeq (X \otimes Y)^\wedge$

- A synthetic spectrum $X \in Syn$ is in the essential image of ν if $\mathbb{1}_{\le 0} \otimes X \in Syn^\heartsuit$.

i.e. $\pi_*^\heartsuit X \simeq \pi_*^\heartsuit \mathbb{1} \underset{\pi_0^\heartsuit \mathbb{1}}{\otimes} \pi_0^\heartsuit X$

Change of notation

1. Syn will mean the ∞-cat. of hypercomplete connective E-based synthetic spectra i.e. $(\widehat{\mathrm{Syn}})_{\geq 0}$.

2. The $(\ )_{\geq 0}$, $(\ )_{\leq 0}$ are w.r.t. $\pi_*^{\heartsuit}$

3. $$\nu : \mathrm{Sp}_E \longrightarrow \mathrm{Syn}$$

The Topological Tower

Consider the Postnikov tower

$$\mathbb{1} \to \cdots \quad \mathbb{1}_{\leq 1} \to \mathbb{1}_{\leq 0}$$

[Pst, 4.54]

$D(\mathrm{Comod}_{E_*E})_{\geq 0}$

$\wr\wr$

$$\rightsquigarrow \quad \mathrm{Syn} \to \quad \cdots \quad \mathrm{Mod}_{\mathbb{1}_{\leq 1}}(\mathrm{Syn}) \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn})$$

pass to module cat.

$\mathrm{Mod}_{C\tau}(\widehat{\mathrm{Syn}})$

$\wr\wr$

$D(\mathrm{Comod})$

<u>Def</u>: A <u>topological potential ℓ-stage</u> is a $\mathbb{1}_{\leq \ell}$-module X in Syn s.t. $\mathbb{1}_{\leq 0} \underset{\mathbb{1}_{\leq \ell}}{\otimes} X$ is discrete.

<u>Notation</u>: $\mathcal{M}_\ell^{top} \underset{\text{full sub.}}{\subseteq} \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$ spanned by topological potential ℓ-stages.

<u>Rem</u>: For $\ell \geq k$, $X \in \mathcal{M}_\ell^{top}$ then extension of scalars defines

$$U_k : \mathcal{M}_\ell^{top} \longrightarrow \mathcal{M}_k^{top}$$

$$X \longmapsto \mathbb{1}_{\leq k} \underset{\mathbb{1}_\ell}{\otimes} X$$

Prop 4.4 : The topological tower

$$\mathcal{M}_\infty^{top} \to \cdots \mathcal{M}_1^{top} \longrightarrow \mathcal{M}_0^{top}$$

has the following properties:

(T1) $\mathcal{M}_\ell^{top}$ is an $(\ell+1)$-category

(T2) $\pi_0 : \mathcal{M}_0^{top} \longrightarrow \mathrm{Comod}_{E_*\hat{E}}$ is an equivalence.

(T3) $X \in \mathcal{M}_{\ell-1}^{top}$, there exists an obstruction in $\mathrm{Ext}^{\ell+2,\ell}(u_0X, u_0X)$ which vanishes iff $\exists \tilde{X} \in \mathcal{M}_\ell^{top}$ s.t. $u_{\ell-1}\tilde{X} \simeq X$.

(T4) There exists fiber sequence

$$\mathrm{Map}_{\mathcal{M}_\ell^{top}}(X,Y) \longrightarrow \mathrm{Map}_{\mathcal{M}_{\ell-1}^{top}}(u_{\ell-1}X, u_{\ell-1}Y)$$
$$\longrightarrow \mathrm{Map}_{\mathcal{D}(\mathrm{Comod}_{E_*\hat{E}})}(u_0X, \Sigma^{\ell+1}u_0Y[-\ell]).$$

(T5) $\mathcal{M}_\infty^{top} \simeq \varprojlim \mathcal{M}_\ell^{top}$

(T6)

$$\begin{tikzcd} \nu : \mathrm{Sp}_E \arrow[r, "\sim"] \arrow[dr, "E_*"'] & \mathcal{M}_\infty^{top} \arrow[d, "u_0"] \\ & \mathcal{M}_0^{top} \simeq \mathrm{Comod}_{E_*E} \end{tikzcd}$$

Algebraic Tower

Recall: $D(E_*E) \simeq \mathrm{Mod}_{P(\mathbb{1})}(D(\mathrm{Comod}_{E_*E}))$ analagous Sp_E

Def: $P := P(\mathbb{1})_{\geq 1}$

$$\pi_* P(\mathbb{1}) = E_*[\tau^{\pm 1}]$$

Fact: The Barnes – Roitzheim algebra $P(\mathbb{1})$ can be recoved as a localization of P, and there is a f.f. embedding

$$\nu_{alg} : \mathrm{Mod}_{P(\mathbb{1})}(D(\mathrm{Comod}_{E_*E})) \longrightarrow \mathrm{Mod}_P(D(\mathrm{Comod}_{E_*E})_{\geq 0})$$

"algebraic synthetic spectra".

- The essential image of ν_{alg} consists of P-modules $P_{\leq 0} \otimes_P M$ is discrete.

Def 4.7

We say that a connective $P_{\leq \ell}$-module M is an

algebraic potential ℓ-stage if $P_{\leq 0} \underset{P}{\otimes} M$ is discrete.

Notation: $\mathcal{M}_\ell^{alg} \underset{\text{full sub.}}{\subseteq} \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$ spanned by

Prop 4.4 : The algebraic tower

$$\mathcal{M}_\infty^{alg} \to \cdots \mathcal{M}_i^{alg} \to \mathcal{M}_0^{alg}$$

has the following properties:

(A1) $\mathcal{M}_\ell^{alg}$ is an $(\ell+1)$-category

(A2) $\pi_0 : \mathcal{M}_0^{alg} \longrightarrow \mathrm{Comod}_{E_*\hat{E}}$ is an equivalence.

(A3) $X \in \mathcal{M}_{\ell-1}^{alg}$, there exists an obstruction in $\mathrm{Ext}^{\ell+2,\ell}(\mathcal{U}_0 X, \mathcal{U}_0 X)$ which vanishes iff $\exists\, \tilde{X} \in \mathcal{M}_\ell^{alg}$ s.t. $\mathcal{U}_{\ell-1}\tilde{X} \simeq X$.

(A4) There exists fiber sequence

$$\mathrm{Map}_{\mathcal{M}_\ell^{alg}}(X, Y) \longrightarrow \mathrm{Map}_{\ell-1}(\mathcal{U}_{\ell-1}X, \mathcal{U}_{\ell-1}Y) \longrightarrow \mathrm{Map}_{\mathcal{D}(\mathrm{Comod}_{E_*\hat{E}})}(\mathcal{U}_0 X, \Sigma^{\ell+1}\mathcal{U}_0 Y[-\ell]).$$

(A5) $\mathcal{M}_\infty^{alg} \simeq \varprojlim \mathcal{M}_\ell^{alg}$

(A6)

$$\begin{tikzcd}
\nu : D(E_*E) \arrow[r, "\sim"] \arrow[dr, "E_*"'] & \mathcal{M}_\infty^{alg} \arrow[d, "\mathcal{U}_0"] \\
& \mathcal{M}_0^{alg} \simeq \mathrm{Comod}_{E_*E}
\end{tikzcd}$$

THM 4.10

Let $p > n+1$. Then we have

$$h_k \mathcal{M}_\infty^{top} \simeq h_k \mathcal{M}_\ell^{top}$$

$$h_k \mathcal{M}_\infty^{alg} \simeq h_k \mathcal{M}_\ell^{alg}$$

In particular, $h_k Sp_E \simeq h_k \mathcal{M}_\ell$ $\qquad k = \ell+1-n^2-n)$

$$h_k D(E_*E) \simeq h_k \mathcal{M}_\ell^{alg}$$

Pf: Since $\mathcal{M}_\infty = \varprojlim_\ell \mathcal{M}_\ell$

Claim: $\ell > n^2+n$,

$u_{\ell'} : \mathcal{M}_{\ell'+1} \longrightarrow \mathcal{M}_{\ell'}$ induces an equivalence

$h_k \mathcal{M}_{\ell'+1} \longrightarrow h_k \mathcal{M}_{\ell'}$ for $\ell' \geq \ell$

Claim: $\ell > n^2+n$,

$U_{\ell'} : \mathcal{M}_{\ell'+1} \longrightarrow \mathcal{M}_{\ell'}$ induces an equivalence

$$h_k \mathcal{M}_{\ell'+1} \longrightarrow h_k \mathcal{M}_{\ell'} \qquad \text{for } \ell' \geq \ell$$

Essential surjectivity:

For $X \in \mathcal{M}_{\ell'}$, by (3)

$$\mathrm{Ext}^{\ell'+3,\, \ell'+1}(U_0 X, U_0 X) = 0$$

b.c. $\ell'+3 \geq \ell+3 \geq n^2+n$. so follows from the vanishing like statement in Comod_{E_*E}.

Fully faithfulness:

By (4), we have a fiber seq.

$$\mathrm{Map}_{\mathcal{M}_{\ell'+1}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{M}_{\ell'}}(U_{\ell'} X, U_{\ell'} Y) \longrightarrow \mathrm{Map}_{D(\mathrm{Comod}_{E_*E})}(U_0 X, \Sigma^{\ell'+2} U_0 Y[-\ell'-1])$$

$$\pi_s(\text{base}) = \mathrm{Ext}^{\ell'+2-s,\, -\ell-1}(U_0 X, U_0 Y) = 0$$

if $\ell^2+2-s > n^2+n$

$\Updownarrow$

$s < \ell'+2-n^2-n = k+1$ $\boxed{z}$

The Bousfield splitting functor & the Bousfield adjunction

1st Goal: Construct a functor

$$\beta : \mathrm{Comod}_{E_*E} \longrightarrow h_k \mathrm{Sp}_E$$

which can be seen as a partial inverse of taking E_*.

2nd Goal: Construct an adjunction

$$\beta^* : D(\mathrm{Comod}_{E_*E})_{\geq 0} \rightleftarrows \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) : \beta_*$$

UPSHOT: β_* is monadic.

THM [Hovey – Strickland]

If E, E' are Landweber exact hmlg theories of the same height. Then

$$\mathrm{Comod}_{E_*E} \simeq \mathrm{Comod}_{E'_*E'}$$

So we can choose E that is the most convenient for us.

We will choose the Johnson – Wilson theory $E(n)$.

Let $q = 2p-2$

UPSHOT: $E(n)_*$, $E(n)_*E(n)$ are concentrated in degrees divisible by q.

Def 2.7 :

Let $q = 2p-2$ and $\varphi \in \mathbb{Z}/q$.

① An E_*E-comodule is pure of phase φ if it is concentrated in degrees $d \equiv \varphi \mod q$.

Notation : $\mathrm{Comod}^{\varphi}_{E_*E}$ is the subcat. of pure comodules of phase φ.

Fact :

$$\mathrm{Comod}_{E_*E} \simeq \prod_{\varphi \in \mathbb{Z}/q} \mathrm{Comod}^{\varphi}_{E_*E}$$

$$\bigoplus_{\varphi \in \mathbb{Z}/q} M^{\varphi} \longleftarrow (M^{\varphi})_{\varphi \in \mathbb{Z}/q}$$

② A spectrum X is pure of phase φ if its homology E_*X is.

We say X is split if it is a finite sum of pure spectra.

Rem : Not every spectrum is split.

THM 2.10 : [G-H Obstruction theory]

For $2p > n^2+n$ and $p > n+1$, Then for any E_*E-comodule M,

there exists a split spectrum X s.t.

$$E_*X \cong M.$$

Lemma 2.12

Suppose $2p-2 > n^2+n$ and let X, Y be pure E-local spectra of the same phase.

Then

$$\mathrm{Map}_{Sp_E}(X, Y) \longrightarrow \mathrm{Hom}_{E_*E}(E_*X, E_*Y)$$

is $(2p-2-n^2-n)$-connected.

Combine these together, one gets:

THM 2.13: $\varphi \in \mathbb{Z}/q$

$E_* : Sp_E \longrightarrow \mathrm{Comod}_{E_*E}$ induces an equivalence

$$h_k Sp_E^{\varphi} \simeq \mathrm{Comod}_{E_*E}^{\varphi} \quad =: R^{\varphi}$$

$k = 2p-2-n^2-n$

Def 2.14: (Bousfield splitting functor)

For $2p-2 > n^2+n$ and $k \leq 2p-3-n^2-n$

Let $\varphi \in \mathbb{Z}/q$, we define $\beta : \mathrm{Comod}_{E_*E} \longrightarrow h_k Sp_E$ as the Composite

$$\mathrm{Comod}_{E_*E} \simeq \prod_{\varphi \in \mathbb{Z}/q} \mathrm{Comod}_{E_*E}^{\varphi} \xrightarrow{\prod R^{\varphi}} \prod_{\varphi \in \mathbb{Z}/q} h_k Sp_E^{\varphi} \xrightarrow{\oplus} h_k Sp_E$$

which we will call the <u>Bousfield splitting functor</u>.

- $E_*(\beta M) \simeq M$
- The essential image of β consists of split spectra.

Bousfield Adjunction

$$\begin{tikzcd}
\mathrm{Sp}_E \arrow[rr, "(\mathbb{1}_{\le e}\otimes -)\circ U"] \arrow[dr] & & \mathrm{Mod}_{\mathbb{1}_{\le e}}(\mathrm{Syn}) \\
& h_{e+1}\mathrm{Sp}_E \arrow[ur, dashed, "(\mathbb{1}_{\le e}\otimes -)\circ U"'] & \\
\mathrm{Comod}^{f.p.}_{E_*E} \arrow[ur, "\beta"] & &
\end{tikzcd}$$

sym. mon.

Thm 2.19.

Lemma 5.2

The composite $(\mathbb{1}_{\le 0}\otimes -)\circ U\circ\beta : \mathrm{Comod}^{fp}_{E_*E} \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\le 0}}(\mathrm{Syn})$

is symm. mon.

Lemma 5.3

The functor $(\mathbb{1}_{\le 0}\otimes -)\circ U\circ\beta : \mathrm{Comod}^{f.p}_{E_*E} \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\le 0}}(\mathrm{Syn})$

uniquely extends to a sym. mon. cocontinuous functor

$$D(\mathrm{Comod}_{E_*E})_{\ge 0} \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\le 0}}(\mathrm{Syn}).$$

Lemma 5.4

The functor $(\mathbb{1}_{\leq \ell} \otimes -) \circ \nu \circ \beta : \mathrm{Comod}^{f.p}_{E_*E} \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$

uniquely extends to a sym. mon. cocontinuous functor

$\beta^* : D(\mathrm{Comod}_{E_*E})_{\geq 0} \longrightarrow \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$.

Apply adjoint functor theorem, one gets.

Def 5.5 (Bousfield Adjunction)

$$\beta^* : D(\mathrm{Comod}_{E_*E})_{\geq 0} \underset{\longleftarrow}{\overset{\dashv}{\longrightarrow}} \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) : \beta_*$$

THM 5.10: β_* is monadic.

THM 6.4: The functor $\beta_* : \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \to D(\mathrm{Comod}_{E_*E})_{\geq 0}$ lifts to a sym. mon. adjoint equivalence.

$$\gamma^* : \mathrm{Mod}_{\beta^*\mathbb{1}_{\leq \ell}}(D(\mathrm{Comod}_{E_*E})_{\geq 0}) \rightleftarrows \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) : \gamma_*$$

Sketch:

b.c. $\mathbb{1}_{\leq \ell}$ is the unit of $\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})$

$$\begin{tikzcd} & \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn})) \arrow[dl, "\simeq"'] \arrow[r] & \mathrm{Mod}_{\beta_*\mathbb{1}_{\leq \ell}}(D(\mathrm{Comod}_{E_*E})_{\geq 0}) \\ \mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \arrow[urr, bend right, "\gamma_*"'] & & \end{tikzcd}$$

Fact: The functor γ_* is lax sym. mon.

2 Claim : γ_* is an equivalence

$$
\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}_{\le \ell}}(\mathrm{Syn}) \arrow[rr, shift left, "\gamma^*"] \arrow[dr, shift left, "\beta_*"] & & \mathrm{Mod}_{\beta_*\mathbb{1}_{\le \ell}}(D(\mathrm{Comod}_{E_*E})_{\ge 0}) \arrow[ll, shift left, "\gamma_*"] \arrow[dl, shift left, "U"] \\
& D(\mathrm{Comod}_{E_*E})_{\ge 0} \arrow[ul, shift left, "\beta^*"] \arrow[ul, phantom, "\perp"] \arrow[ur, shift left, "(\beta_*\mathbb{1}_{\le \ell}\otimes -)"] &
\end{tikzcd}
$$

$U \leftarrow$ forget fnl

Since U is monadic, by [HA 4.7.3.16] it will suffice to show for an $M \in D(\mathrm{Comod}_{E_*E})$

$$\beta_*\mathbb{1}_{\le \ell}\otimes M \longrightarrow \beta_*\beta^*M$$

is an equivalence.

3 Claim : γ^* is sym. mon.

...

$\boxed{?}$

Finishing the proof:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{1}_{\leq \ell}}(\mathrm{Syn}) \arrow[r, "\gamma^*", "\simeq"'] \arrow[d, "\mathbb{1}_{\leq 0} \otimes_{\mathbb{1}_{\ell}} (-)"'] & \mathrm{Mod}_{\beta_* \mathbb{1}_{\leq \ell}}(D(\mathrm{Comod}_{E_*E})_{\geq 0}) \arrow[d, "\beta_* \mathbb{1}_{\leq 0} \otimes_{\beta_* \mathbb{1}_{\ell}} (-)"] \\
\mathrm{Mod}_{\mathbb{1}_{\leq 0}}(\mathrm{Syn}) \arrow[r, "\simeq"] & \mathrm{Mod}_{\beta_* \mathbb{1}_{\leq 0}}(D(\mathrm{Comod}_{E_*E})_{\geq 0})
\end{tikzcd} \quad (*)$$

Crux: We have a comm. diagram

this is an equivalence as associative algebras.

$$\begin{tikzcd}
\beta_* \mathbb{1}_{\leq \ell} \arrow[r, "\simeq"] \arrow[d] & P_{\leq \ell} \arrow[d] \\
(\beta_* \mathbb{1}_{\leq \ell}) \arrow[r] & P_{\leq 0}
\end{tikzcd}$$

This concludes the proof of the main theore, the commutativity of $(*)$ would imply γ^* restricts to an equivalence

$$\mathcal{M}_{\ell}^{top} \xrightarrow{\simeq} \mathcal{M}_{\ell}^{alg}.$$

$\square$