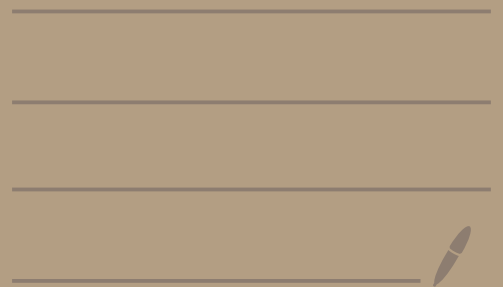


A synthetic Toda bracket



Review:

Thm 1.4. Let $M\langle 4n \rangle$ be $\text{Thom}(\tau_{\geq 4n} B\mathbb{O} \rightarrow B\mathbb{O})$

For $n \geq 3$

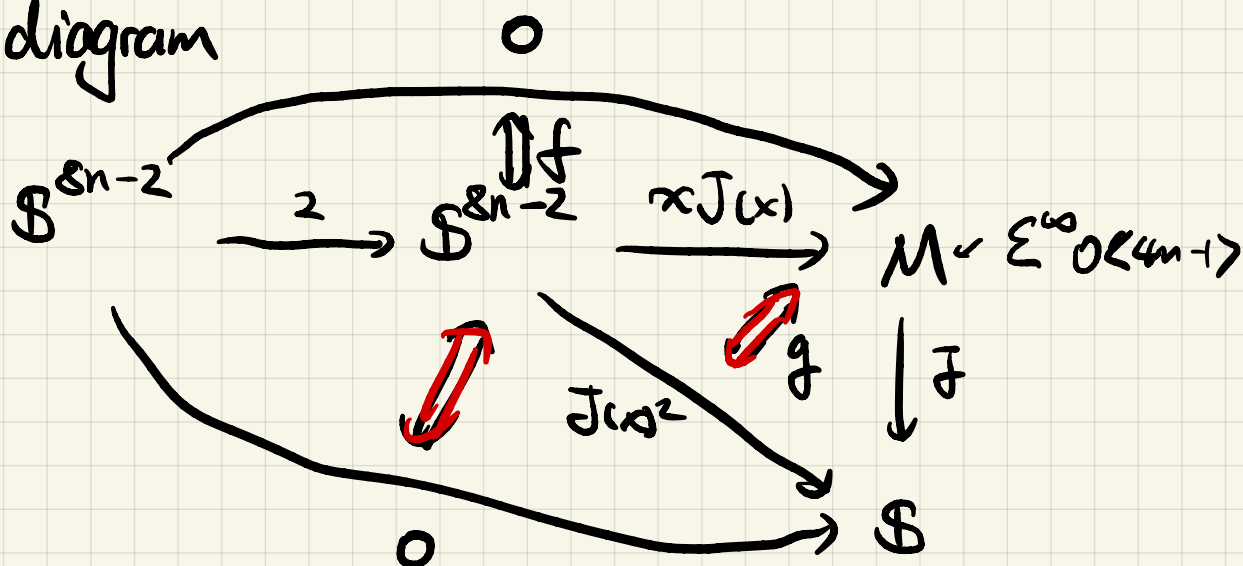
$\pi_{8n-1} \mathbb{S} \rightarrow \pi_{8n-1} M\langle 4n \rangle$
 is surj. with ker. exactly $(\text{im } J)_{8n-1}$.

Yujing's talk: $M\langle 4n \rangle = | \text{Bor}(\mathbb{S}, \Sigma_+^{\infty} \mathbb{O}\langle 4n-1 \rangle, \mathbb{S}) |$.

In $\mathbb{O}\langle 4n-1 \rangle$.

Gijs' talk: $M \hookrightarrow \Sigma^{\infty} \mathbb{O}\langle 4n-1 \rangle$ be the $(8n-1)$ -skeleton
 $\times \in \pi_{4n-1}(\Sigma^{\infty} \mathbb{O}\langle 4n-1 \rangle)$

diagram



$w \in \pi_{8n-1} \mathbb{S}$

If we can choose f s.t. $w \in \text{im } J_{8n-1}$ then we are done.

① Thm 10.8 $\exists f$ s.t. w has Adams filtration $\geq 2N_p - 1$

$$N_2 := h(4n-1) - \lfloor \log_2(8n) \rfloor + 1$$

$$h(s) := |\{t \mid 0 < t \leq s, t \equiv 0, 1, 2, 4 \pmod{8}\}|.$$

$$N_p := \left\lfloor \frac{4n}{2p-2} \right\rfloor - \lfloor \log_p(4n) \rfloor.$$

②

For $n \geq 31$ such w lies in $\text{im } J_{E, n-2}$.

Review of synthetic spectra.

A stable presentably SM ω -cat Syn_E
for Adams type homology theory E .

$v_E: \text{Sp} \rightarrow \text{Syn}_E$ lax monoidal.

$E = \text{HF}_p$ sym. monoidal.

Idea: $v_E(X)$ records the complete info. about E -ASS of X .

$E = \text{HF}_p$

- bigraded htpy in Syn_E

- an element τ .

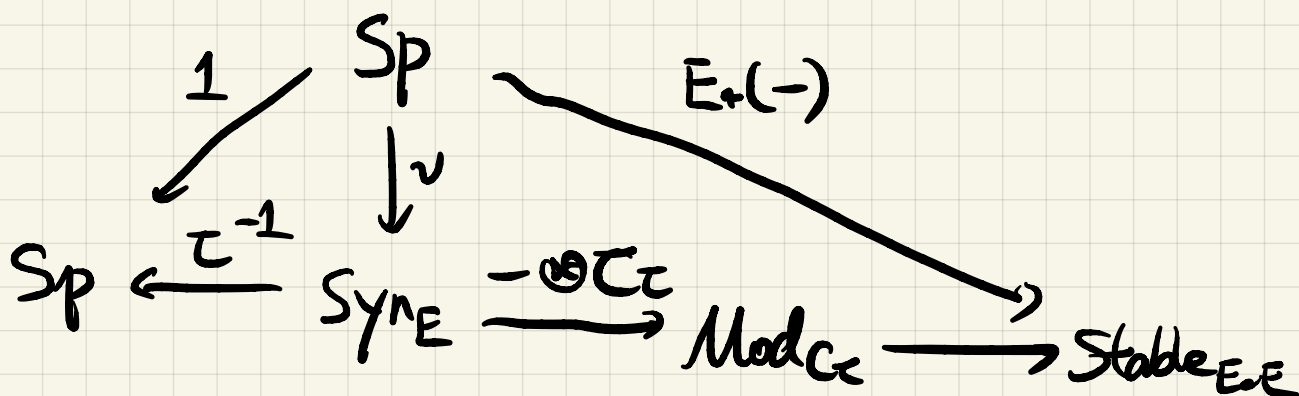
Def. $\mathcal{S}^{n,n} := \nu \mathcal{S}^n$ $\mathcal{S}^{a,b} := \sum \mathcal{S}^{a-b, b}$

$\pi_{ab}(x) := \pi_0 \text{Hom}(\mathcal{S}^{a,b}, x)$

$\tau: \mathcal{S}^{0,-1} = \sum \nu(\mathcal{S}^{-1}) \rightarrow \nu(\sum \mathcal{S}^{-1}) = \mathcal{S}^{0,0}$

Thm 9.12

- (1) $\tau^{-1}: \text{Syn}_E \rightarrow \text{Syn}_E$ is sym. monoidal.
- (2) τ -invertible syn. sp \simeq Sp.
- (3) $\tau^{-1} \circ \nu \simeq \text{id}_{\text{Sp}}$
- (4) $\mathbb{C}\tau$ is an E_0 -ring in Syn_E .
- (5) $\text{Mod}_{\mathbb{C}\tau} \rightarrow \text{Stable}_{E \rightarrow E}$ s.t.



Cor. 9.13

$\pi_{t-s,t}(\mathbb{C}\tau \otimes \nu X) \simeq \text{Ext}_{E \rightarrow E}^{s,t}(E_-, E_- X)$

\uparrow
 E_2 of E -ASS of X .

Lemma 9.15 $f: X \rightarrow Y$ has E-ASS fil. $\geq k$.

then there is a factorization

$$\begin{array}{ccc} \tilde{f} & \xrightarrow{\quad} & \Sigma^{0, -k} \cup Y \\ & \searrow & \downarrow \tau^k \\ \nu X & \xrightarrow{\nu f} & \nu Y \end{array}$$

Thm 9.19 X is E-nilpotent complete with strongly conv. E-ASS.

κ is a class in stem k , filtration s of E_2 -page.

TFAE:

(1a) d_2, \dots, d_r vanishes on κ .

(1b) $\kappa \in \pi_{k, k+s}(\mathbb{C}\tau \otimes \nu X)$ lifts to $\pi_{k, k+s}(\mathbb{C}\tau^s \otimes \nu X)$

(1c) κ has such a lift that

$$\begin{array}{ccc} \mathbb{C}\tau^r \otimes \nu X & \longrightarrow & \Sigma^{1, -r} \mathbb{C}\tau \otimes \nu X \\ \tilde{\kappa} & \longmapsto & -d_{r+1}(\kappa) \end{array}$$

If κ is a p.c. \exists lift κ to $\tilde{\kappa} \in \pi_{k, k+s}(\nu X)$

with

(2a) If κ survives to E_{r+1} -page, then $\tau^{r-1}\tilde{\kappa} \neq 0$.

(2b) If κ survives to E_∞ -page then $\tilde{\kappa} \in \pi_{k, k+s}(\tau^{-1} \otimes \nu X)$

is of E-Adams filt. s and detects κ .

We can choose a lift $\tilde{\alpha} \in \pi_{k, k+s}(vX)$ s.t.

(3a) if α is killed by d_{r+1} $\tau^r \tilde{\alpha} = 0$.

(3b) if α survives to E_{∞} , $\alpha \in \pi_{k, k}$ detected by α .
then we can choose $\tilde{\alpha}$ s.t. $\tau^{-1}(\tilde{\alpha}) = \alpha$.

(4) Fix a collection of $\tilde{\alpha}$ s.t. image α in C_{τ} .

Spans p.c. in top. deg. k . Then τ -completion
of subgp. of $\pi_{k, *}(vX)$ gen. by $\tilde{\alpha}$ is $\pi_{k, *}(vX)$.

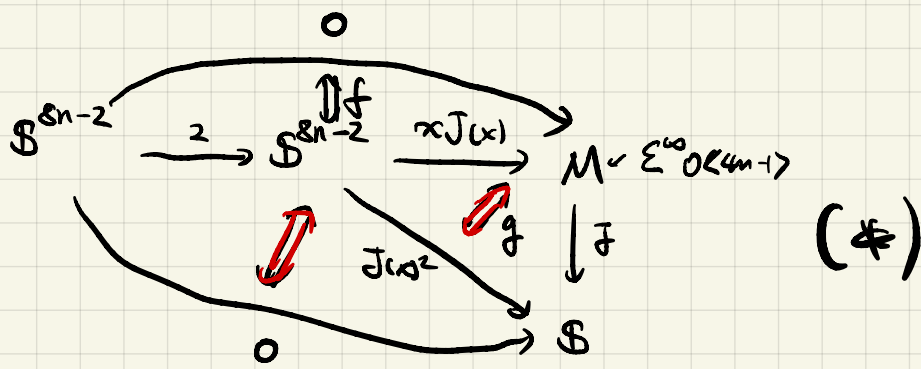
Cor. 9.20. X above for $a, b \forall s \geq 0 \pi_{a, b+s}(C_{\tau} \otimes vX) = 0$

then $\pi_{a, b+s}(vX) = 0$ for all $s \geq 0$.

Cor. 9.21 Consider the filtration of $\pi_k(X)$ by

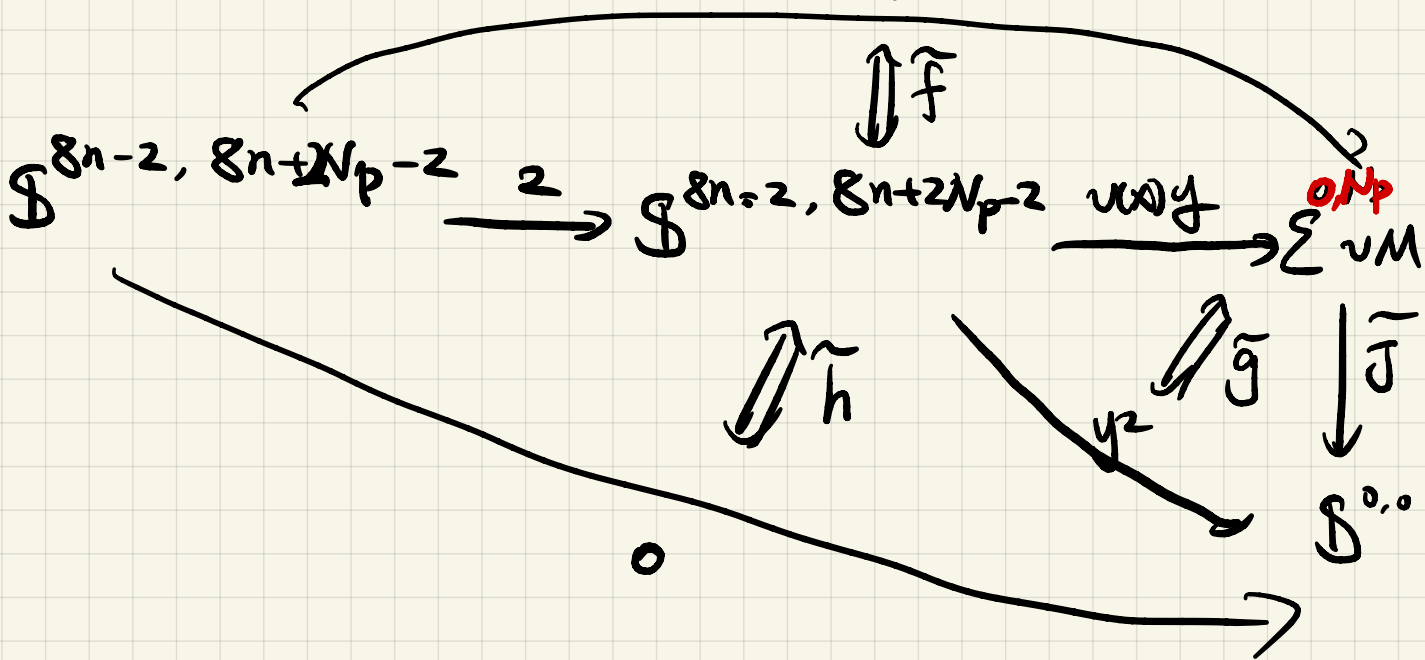
$$F^s \pi_k(X) := \text{im} \left(\pi_{k, k+s}(vX) \xrightarrow{\tau^{-1}} \pi_k(X) \right)$$

it is the E-Adams filtration.



w has high Adams filtration

Thm. There is a diagram in $\text{Syn}_{\text{HFF}_p}$.



$$\tau^{-1}(\text{diagram}) = \text{diagram } (*).$$

$$\hat{w} \in \pi_{8n-1, 8n+2N_p-2} S^{0,0} \implies w = \tau^{-1}(\hat{w})$$

has Adams fil $\geq 2N_p - 2$

• Lemma 10.4. $M \rightarrow \Sigma^{\infty} O\langle 4n-1 \rangle \xrightarrow{\tilde{f}} \mathcal{S}$
 has \mathbb{F}_p -Adams filtration at least N_p .

① lift \tilde{f} to \tilde{f}

② $y: \mathcal{S}^{4n-1, 4n+N_p-1} \xrightarrow{\nu(x)} \Sigma^{0, N_p} \nu M \xrightarrow{\tilde{f}} \mathcal{S}^{0,0}$.

• Prop. 10.7. $\pi_{8n-2, 8n+N_p-2}(\nu M) = 0$ for $n \geq 3$.

\Downarrow
 the homotopy \tilde{f}

- Lemma 10.4. $M \rightarrow \Sigma^\infty O\langle 4n-1 \rangle \xrightarrow{J} \mathcal{S}$
has \mathbb{F}_p -Adams filtration at least N_p .

Idea: Factor this map through at least N_p maps of Adams filtration ≥ 1 .

- $f: X \rightarrow Y$ has Adams fil $\geq 1 \iff H^*(Y) \rightarrow H^*(X)$ is zero.

$$\begin{array}{c}
 p=2: \quad \Sigma^\infty O\langle 4n-1 \rangle \\
 \downarrow \\
 \Sigma^\infty O\langle 4n-2 \rangle \\
 \downarrow \\
 \vdots \\
 \downarrow \\
 \Sigma^\infty O\langle 1 \rangle \cong \Sigma^\infty SO \xrightarrow{J} \mathcal{S}
 \end{array}
 \left. \vphantom{\begin{array}{c} \Sigma^\infty O\langle 4n-1 \rangle \\ \Sigma^\infty O\langle 4n-2 \rangle \\ \vdots \\ \Sigma^\infty O\langle 1 \rangle \end{array}} \right\} \text{has Adams fil } \geq 1$$

Hope: restricting to $M: (8n-1)$ -skeleton a lot of maps are 0 in H^* .

i.e. $\Sigma^\infty O\langle m \rangle \rightarrow \Sigma^\infty O\langle m-1 \rangle$ induces 0 on $H^*(-)$ for $* \leq 8n-1$.

Cor. 10.15. Assume $m \equiv 0, 1, 2, 4 \pmod{8}$.

Then $O\langle m \rangle \rightarrow O\langle m-1 \rangle$ induces 0 on $H^*(-)$ for $0 < * < 2^{h(m)} - 1$.

Cor. 10.15. Assume $m \equiv 0, 1, 2, 4 \pmod 8$.

Then $O\langle m \rangle \rightarrow O\langle m-1 \rangle$ induces 0 on $H^*(-)$
for $0 < * < 2^{h(m)} - 1$.

Lemma. 10.18 $M_k \rightarrow \Sigma^\infty O\langle m-1 \rangle$ be k -skeleton.

Then $M_k \rightarrow \Sigma^\infty O\langle m-1 \rangle \xrightarrow{\tilde{J}} \mathcal{S}$ has \mathbb{F}_2 -Adams
fil $\geq h(m-1) - \lfloor \log_2(k+1) \rfloor + 1$.

[Take $m=4n$ $k=8n-1$ RHS = N_2]

Pf of 10.18: Adams fil. $\geq |\{s \in \mathbb{N} \mid s \equiv 0, 1, 2, 4 \pmod 8$

$$2 \leq s \leq m-1,$$

$$k < 2^{h(m)} - 1 \rfloor + 1$$

\uparrow

$$\Sigma^\infty O\langle m-1 \rangle \xrightarrow{\tilde{J}} \mathcal{S}$$

has Adams fil. ≥ 1

||

$$h(m-1) - \lfloor \log_2(k+1) \rfloor + 1 \quad \square$$

Sketch of pf of 10.15:

$$\begin{array}{ccc}
 O\langle m \rangle & & \pi_m BO \\
 \downarrow & & \parallel \\
 & & \pi_m BO\langle m \rangle \\
 & & \parallel \\
 O\langle m-1 \rangle & \xrightarrow{\varphi} & K(\pi_{m-1} O\langle m-1 \rangle, m-1)
 \end{array}$$

Hope: φ^* is surj on $H^*(-)$ for $*$ $< 2^{h(m)} - 1$.

Compare EMSS

$$\begin{array}{ccc}
 O\langle m-1 \rangle \rightarrow * & & K(\pi_m BO, m-1) \rightarrow * \\
 \downarrow & \xrightarrow{\varphi} & \downarrow \\
 BO\langle m \rangle & & K(\pi_m BO, m)
 \end{array}$$

① Both collapse at E_2 [collapsing thm].

$$\text{Tor}_{H^*(\text{Base})}^{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H^*(\text{fibre})$$

② In our range. $H^*(\text{fibre})$ are gen. by Tor^1 of the base.

③ $BO\langle m \rangle \xrightarrow{\varphi} K(\pi_m BO, m)$ is surj. on $H^*(-; \mathbb{F}_2)$
for $*$ $< 2^{h(m)}$

② and ③ Strong: $H^*(BO\langle m \rangle; \mathbb{F}_2) \cong \text{Poly alg. } \otimes \text{im } \varphi^*$
with gen $\cong h(m)$ \uparrow poly. alg.

□

p odd: analogous computation on $U\langle n \rangle$ $O\langle n \rangle \hookrightarrow U\langle n \rangle$
is a summand.

• Prop. 10.7. $\pi_{8n-2, 8n+Np-2}(vM) = 0$ for $n \geq 3$.

Pf: E_2 -page is 0 above the filtration Np .

$$\pi_{8n-2, 8n+k-2}(vM) = 0$$

\uparrow 9.13

$$\pi_{8n-2, 8n+k-2}(C\tau \otimes vM) = 0$$

\Downarrow

$$E_2^{k, 8n-2+k}(M) = 0 \quad \pi_{8n-2} \cong \mathbb{Z}/2$$

$$\pi_{8n-1} \cong 0.$$

\downarrow

• Goodwillie tower:

$$D_2(\Sigma^{-1} \tau_{\geq 4n} k_0) \quad \Sigma^{-1} \tau_{\geq 4n} k_0$$

\downarrow

\downarrow s1

$$\Sigma^\infty \langle 0 < 4n-1 \rangle \rightarrow \dots \rightarrow Q_2 \rightarrow Q_1$$

p odd: only $\Sigma^{-1} \tau_{\geq 4n} k_0$ matters

$$E_2^{k, 8n-2+k}(\Sigma^{-1} \tau_{\geq 4n} k_0) = 0 \text{ for } k \geq 0.$$

• $\tau_{\geq 4n} k_0 \simeq \Sigma^{4n} k_0$ in our range. • p odd $k_0 \hookrightarrow k_{4n}$ as a summand.

$$E_2^{h, 8n-2+k}(k_{4n}) = 0 \text{ for } k \geq 0 \text{ Green book.}$$

$p=2$: Prop 10.21: $n \geq 3$ in $t-s \leq 8n-3$
there is an iso.

$$E_2^{s,t}(\Sigma^\infty O\langle 4n-1 \rangle) \cong E_2^{s,t}(\Sigma^{-1} \mathcal{L}_{\geq 4n} k_0)$$

$$t-s = 8n-2$$

$$E_2^{s,t}(\Sigma^\infty O\langle 4n-1 \rangle) \cong E_2^{s-1,t}(\Sigma D_2(\Sigma^{-1} \mathcal{L}_{\geq 4n} k_0))$$

$$\cong \begin{cases} 0 & (s,t) \neq (1, 8n-1) \\ \mathbb{Z}/2 & (s,t) = (1, 8n-1) \end{cases}$$

Sketch of Pf:

- $\Sigma^\infty O\langle 4n-1 \rangle \rightarrow \Sigma^{-1} \mathcal{L}_{\geq 4n} k_0$
is surj. on $H^*(-)$ for $*$ $\leq 8n-1$ when $n \geq 3$.

$$\begin{array}{ccc} \Sigma^\infty O\langle 4n-1 \rangle & \longrightarrow & \Sigma^{-1} \mathcal{L}_{\geq 4n} k_0 \\ \text{surj-in range} \rightsquigarrow \downarrow \varphi & & \downarrow \\ \Sigma^\infty K(\mathbb{Z}, 4n-1) & \xrightarrow{\quad} & \Sigma^{4n-1} H\mathbb{Z} \\ \downarrow \text{surj:} & \text{Serre} & \end{array}$$

SES:

$$0 \rightarrow H^*(E_2(\Sigma^{-1} \tau_{\geq 4n} h_0)) \rightarrow H^*(\Sigma^{-1} \tau_{\geq 4n} h_0)$$

\uparrow
Ext $_A$ has $\mathbb{Z}/2$
in $(0, 8n-1)$

goes to $(1, 8n-1)$

for $s \leq 8n-1$. In $E_2(0 < 4n-1)$

$$\downarrow$$
$$H^*(0 < 4n-1)$$

\downarrow
0

\nearrow
contribute to
 $t-s \leq 8n-3$.

nothing at $t-s = 8n-2$.

□