A synthetic Toda bracket

Review:

Thm 1.4. Let $MO\langle 4n\rangle$ be $\mathrm{Thom}(\tau_{\geq 4n}BO \to BO)$

For $n \geq 31$

$$\pi_{8n-1}\mathbb{S} \longrightarrow \pi_{8n-1} MO\langle 4n\rangle$$

is surj. with ker. exactly $(\mathrm{im} J)_{8n-1}$.

Yuqing's talk: $MO\langle 4n\rangle = |\mathrm{Bar}(\mathbb{S}, \Sigma^\infty_+ O\langle 4n-1\rangle, \mathbb{S})_\bullet|$.

In $O\langle 4n-1\rangle$.

Gijs' talk: $M \hookrightarrow \Sigma^\infty O\langle 4n-1\rangle$ be the $(8n-1)$-skeleton

$x \in \pi_{4n-1}(\Sigma^\infty O\langle 4n-1\rangle)$

diagram

$$\begin{tikzcd}
\mathbb{S}^{8n-2} \arrow[r, "2"] \arrow[rr, bend left=50, "0"] \arrow[rrd, bend right=50, "0"'] & \mathbb{S}^{8n-2} \arrow[u, phantom, "\Updownarrow f"] \arrow[r, "xJ(x)"] \arrow[rd, "J(x)^2"'] & M \subset \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J"] \arrow[ld, phantom, "g"] \\
& & \mathbb{S}
\end{tikzcd}$$

$w \in \pi_{8n-1}\mathbb{S}$

If we can choose f s.t. $w \in \mathrm{im} J_{8n-1}$ then we are done.

① Thm 10.8 $\exists f$ s.t. w has Adams filtration $\geq 2N_p - 1$

$$N_2 := h(4n-1) - \lfloor \log_2(8n) \rfloor + 1$$

$$h(s) := |\{t \mid 0 < t \leq s,\ t \equiv 0,1,2,4 \bmod 8\}|$$

$$N_p := \left\lfloor \frac{4n}{2p-2} \right\rfloor - \lfloor \log_p(4n) \rfloor.$$

②

For $n \geq 31$ such w lies in $\mathrm{im} J_{8n-1}$.

Review of synthetic spectra.

A stable presentably SM ∞-cat Syn_E for Adams type homology theory E.

$\nu_E : \mathrm{Sp} \longrightarrow \mathrm{Syn}_E$ lax monoidal.

$E = H\mathbb{F}_p$ sym. monoidal.

Idea: $\nu_E(X)$ records the complete info. about E-ASS of X.

$E = H\mathbb{F}_p$

- bigraded htpy in Syn_E
- an element τ.

Def. $\mathbb{S}^{n,n} := \nu\mathbb{S}^{n}$ $\qquad \mathbb{S}^{a,b} := \Sigma^{a-b}\mathbb{S}^{b,b}$.

$$\pi_{a,b}(X) := \pi_0 \operatorname{Hom}(\mathbb{S}^{a,b}, X).$$

$$\tau : \mathbb{S}^{0,-1} = \Sigma\nu(\mathbb{S}^{-1}) \longrightarrow \nu(\Sigma\mathbb{S}^{-1}) = \mathbb{S}^{0,0}$$

Thm 9.12

(1) $\tau^{-1} : \mathrm{Syn}_E \longrightarrow \mathrm{Syn}_E$ is sym. monoidal.

(2) τ-invertible syn. sp $\simeq$ Sp.

(3) $\tau^{-1} \circ \nu \simeq \mathrm{id}_{\mathrm{Sp}}$

(4) $C\tau$ is an E_∞-ring in Syn_E.

(5) $\mathrm{Mod}_{C\tau} \longrightarrow \mathrm{Stable}_{E_*E}$ s.t.

$$\begin{tikzcd}
 & \mathrm{Sp} \arrow[dl, "1"'] \arrow[d, "\nu"] \arrow[drr, bend left, "E_*(-)"] & & \\
\mathrm{Sp} & \mathrm{Syn}_E \arrow[l, "\tau^{-1}"'] \arrow[r, "-\otimes C\tau"] & \mathrm{Mod}_{C\tau} \arrow[r] & \mathrm{Stable}_{E_*E}
\end{tikzcd}$$

Cor. 9.13

$$\pi_{t-s,t}(C\tau \otimes \nu X) \simeq \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X)$$

E_2 of E-ASS of X.

Lemma 9.15 $f: X \to Y$ has E-ASS fil. $\geq k$.

then there is a factorization

$$\begin{tikzcd} & & \Sigma^{0,-k} \nu Y \arrow[d, "\tau^k"] \\ \nu X \arrow[urr, dashed, "\tilde{f}"] \arrow[rr, "\nu f"] & & \nu Y \end{tikzcd}$$

Thm 9.19 X is E-nilpotent complete with strongly conv. E-ASS.

x is a class in stem k, filtation s of E_2-page.

TFAE:

(1a) $d_2, \cdots, d_r$ vanishes on x.

(1b) $x \in \pi_{k,k+s}(C\tau \otimes \nu X)$ lifts to $\pi_{k,k+s}(C\tau^r \otimes \nu X)$

(1c) x has such a lift that

$$\begin{aligned} C\tau^r \otimes \nu X &\longrightarrow \Sigma^{1,-r} C\tau \otimes \nu X \\ \tilde{x} &\longmapsto -d_{r+1}(x) \end{aligned}$$

If x is a p.c. $\exists$ lift x to $\tilde{x} \in \pi_{k,k+s}(\nu X)$ with

(2a) If x survives to E_{r+1}-page, then $\tau^{r-1}\tilde{x} \neq 0$.

(2b) If x survives to E_∞-page then $\tilde{x} \in \pi_k(\underset{\pi_k(X)}{\overset{}{\tau^{-1}\nu X}})$

is of E-Adams filt. s and detects x.

We can choose a lift $\hat{x} \in \pi_{k,k+s}(\nu X)$ s.t.

(3a) if x is killed by d_{r+1} $\tau^r \hat{x} = 0$.

(3b) if x survives to E_∞. $\alpha \in \pi_k X$ detected by x.
then we can choose $\hat{x}$ s.t. $\tau^{-1}(\hat{x}) = \alpha$.

(4) Fix a collection of $\hat{x}$ s.t. image x in $C\tau$.
Spans p.c. in top. deg. k. Then τ-completion of subgp. of $\pi_{k,*}(\nu X)$ gen. by $\hat{x}$ is $\pi_{k,*}(\nu X)$.

Cor. 9.20. X above for a, b $\forall s \geq 0$ $\pi_{a,b+s}(C\tau \otimes \nu X) = 0$
then $\pi_{a,b+s}(\nu X) = 0$ for all $s \geq 0$.

Cor. 9.21 Consider the filtration of $\pi_k(X)$ by

$$F^s \pi_k(X) := \mathrm{im}\left(\pi_{k,k+s}(\nu X) \xrightarrow{\tau^{-1}} \pi_k(X) \right)$$

it is the E-Adams filtration.

$$
\begin{tikzcd}
\mathbb{S}^{8n-2} \arrow[r, "2"] \arrow[rr, bend left=50, "0"] \arrow[rrd, bend right=40, "0"'] & \mathbb{S}^{8n-2} \arrow[r, "xJ(x)"] \arrow[dr, "J(x)^2"'] \arrow[r, phantom, bend left=30, "\Updownarrow f"] \arrow[d, phantom, "\Updownarrow"] & M = \Sigma^\infty O\langle 4n-1\rangle \arrow[d, "J"] \arrow[dl, phantom, "\Updownarrow g"] \\
& & \mathbb{S}
\end{tikzcd}
\qquad (*)
$$

w has high Adams filtration

Thm. There is a diagram in $\mathrm{Syn}_{H\mathbb{F}_p}$.

$$
\begin{tikzcd}
\mathbb{S}^{8n-2,\,8n+2N_p-2} \arrow[r, "2"] \arrow[rr, bend left=40, "0"] \arrow[rrd, bend right=30, "0"'] & \mathbb{S}^{8n-2,\,8n+2N_p-2} \arrow[r, "\nu(x)y"] \arrow[dr, "y^2"'] \arrow[r, phantom, bend left=30, "\Updownarrow \tilde f"] \arrow[d, phantom, "\Updownarrow \tilde h"] & \Sigma^{0,N_p}\nu M \arrow[d, "\tilde J"] \arrow[dl, phantom, "\Updownarrow \tilde g"] \\
& & \mathbb{S}^{0,0}
\end{tikzcd}
$$

$\tau^{-1}(\text{diagram}) = \text{diagram } (*)$.

$\tilde w \in \pi_{8n-1,\,8n+2N_p-2}\,\mathbb{S}^{0,0} \Longrightarrow w = \tau^{-1}(\tilde w)$

has Adams fil $\geq 2N_p - 2$

• Lemma 10.4. $M \longrightarrow \Sigma^\infty O\langle 4n-1\rangle \xrightarrow{J} \mathbb{S}$

has $\mathbb{F}_p$-Adam filtration at least N_p.

① lift J to $\tilde{J}$

② $y: \mathbb{S}^{4n-1,\, 4n+N_p-1} \xrightarrow{\nu(x)} \Sigma^{0, N_p} \nu M \xrightarrow{\tilde{J}} \mathbb{S}^{0,0}$.

• Prop. 10.7. $\pi_{8n-2,\, 8n+N_p-2}(\nu M) = 0$ for $n \geq 3$.

$\Downarrow$

the homotopy $\tilde{f}$

• Lemma 10.4. $M \longrightarrow \Sigma^\infty O\langle 4n-1\rangle \xrightarrow{J} \mathbb{S}$

has $\mathbb{F}_p$-Adam filtration at least N_p.

Idea: Factor this map through at least N_p maps of Adams filtration ≥ 1.

• $f: X \to Y$ has Adams fil $\geq 1 \iff H^*(Y) \to H^*(X)$ is zero.

$p=2$:

$$\begin{array}{c} \Sigma^\infty O\langle 4n-1\rangle \\ \downarrow \\ \Sigma^\infty O\langle 4n-2\rangle \\ \downarrow \\ \vdots \\ \downarrow \\ \Sigma^\infty O\langle 1\rangle \simeq \Sigma^\infty SO \xrightarrow{J} \mathbb{S} \end{array}$$

has Adam fil ≥ 1

Hope: restricting to M: $(8n-1)$-skeleton a lot of maps are 0 in H^*.

i.e. $\Sigma^\infty O\langle m\rangle \to \Sigma^\infty O\langle m-1\rangle$ induces 0 on $H^*(-)$, for $* \leq 8n-1$.

Cor. 10.15. Assume $m \equiv 0, 1, 2, 4 \mod 8$.

Then $O\langle m\rangle \longrightarrow O\langle m-1\rangle$ induces 0 on $H^*(-)$

for $0 < * < 2^{h(m)} - 1$.

Cor. 10.15. Assume $m \equiv 0, 1, 2, 4 \mod 8$.
Then $O\langle m\rangle \longrightarrow O\langle m-1\rangle$ induces 0 on $H^*(-)$
for $0 < * < 2^{h(m)} - 1$.

Lemma. 10.18 $M_k \longrightarrow \Sigma^\infty O\langle m-1\rangle$ be k-skeleton.
Then $M_k \longrightarrow \Sigma^\infty O\langle m-1\rangle \xrightarrow{J} \mathbb{S}$ has $\mathbb{F}_2$-Adams
fil $\geq h(m-1) - \lfloor \log_2(k+1) \rfloor + 1$.

[Take $m = 4n$ $k = 8n-1$ RHS $= \nu_2$]

Pf of 10.18:

$$\text{Adams fil.} \geq \left|\left\{ s \in \mathbb{N} \;\middle|\; \begin{array}{l} s \equiv 0, 1, 2, 4 \mod 8 \\ 2 \leq s \leq m-1, \\ k < 2^{h(s)} - 1 \end{array} \right\}\right| + \underset{\substack{\uparrow \\ \Sigma^\infty SO \xrightarrow{J} \mathbb{S} \\ \text{has Ad. fil} \geq 1}}{1}$$

$$\| $$

$$h(m-1) - \lfloor \log_2(k+1) \rfloor + 1 \quad \square$$

Sketch of pf of 10.15:

$$\begin{array}{ccc} O\langle m\rangle & & \pi_m BO \\ & & \cong \\ \downarrow & & \pi_m BO\langle m\rangle \\ & & \cong \\ O\langle m-1\rangle & \xrightarrow{\varphi} & K(\pi_{m-1} O\langle m-1\rangle, m-1) \end{array}$$

Hope: φ^* is surj on $H^*(-)$ for $* < 2^{h(m)} - 1$.

Compare EMSS

$$\begin{array}{ccc} O\langle m-1\rangle \to * & & K(\pi_m BO, m-1) \to * \\ \downarrow & \xrightarrow{\varphi} & \downarrow \\ BO\langle m\rangle & & K(\pi_m BO, m) \end{array}$$

① Both collapse at E_2 [collapsing thm].

$$\mathrm{Tor}^{*,*}_{H^*(\text{Base})}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H^*(\text{fibre})$$

② In our range. $H^*(\text{fibre})$ are gen. by Tor^1 of the base.

③ $BO\langle m\rangle \xrightarrow{\varphi} K(\pi_m BO, m)$ is surj. on $H^*(-;\mathbb{F}_2)$ for $* < 2^{h(m)}$

② and ③ Strong: $H^*(BO\langle m\rangle;\mathbb{F}_2) \cong$ Poly alg. $\otimes$ $\mathrm{im}\,\varphi^*$

with gen $\geq h(m)$ ↑ poly. alg.

$\square$

p odd: analogous computat. on $U\langle n\rangle$ $O\langle n\rangle \hookrightarrow U\langle n\rangle$ is a summand.

• Prop. 10.7. $\pi_{8n-2,\, 8n+N_p-2}(\nu M) = 0$ for $n \geq 3$.

Pf: E_2-page is 0 above the filtration N_p.

$$\pi_{8n-2,\, 8n+k-2}(\nu M) = 0$$

$$\Uparrow \; 9.13$$

$$\pi_{8n-2,\, 8n+k-2}(C\tau \otimes \nu M) = 0$$

$$\Updownarrow$$

$$E_2^{k,\, 8n-2+k}(M) = 0 \qquad \pi_{8n-2} \cong \mathbb{Z}/2$$

$$\pi_{8n-1} \cong 0.$$

· Goodwillie tower:

$$\begin{tikzcd}
 & & D_2(\Sigma^{-1}\tau_{\geq 4n}ko) \arrow[d] & \Sigma^{-1}\tau_{\geq 4n}ko \arrow[d, "\wr"] \\
\Sigma^\infty O\langle 4n-1\rangle \arrow[r] & \cdots \arrow[r] & Q_2 \arrow[r] & Q_1
\end{tikzcd}$$

p odd: only $\Sigma^{-1}\tau_{\geq 4n}ko$ matters

$$E_2^{k,\, 8n-2+k}(\Sigma^{-1}\tau_{\geq 4n}ko) = 0 \text{ for } k \geq 0.$$

• $\tau_{\geq 4n}ko \simeq \Sigma^{4n}ko$ in our range. • p odd $ko \hookrightarrow ku$ as a summand.

• $E_2^{k,\, 8n-2+k}(ku) = 0$ for $k \geq 0$ Green book.

$p=2$: Prop 10.21 : $n \geq 3$ in $t-s \leq 8n-3$
there is an iso.

$$E_2^{s,t}(\Sigma^\infty O\langle 4n-1\rangle) \cong E_2^{s,t}(\Sigma^{-1}\tau_{\geq 4n}ko)$$

$t-s = 8n-2$

$$E_2^{s,t}(\Sigma^\infty O\langle 4n-1\rangle) \cong E_2^{s-1,t}(\Sigma D_2(\Sigma^{-1}\tau_{\geq 4n}ko))$$

$$\cong \begin{cases} 0 & (s,t) \neq (1, 8n-1) \\ \mathbb{Z}/2 & (s,t) = (1, 8n-1) \end{cases}$$

Sketch of Pf:

- $\Sigma^\infty O\langle 4n-1\rangle \longrightarrow \Sigma^{-1}\tau_{\geq 4n}ko$
  is surj. on $H^*(-)$ for $* \leq 8n-1$ when $n \geq 3$.

$$\begin{tikzcd} \Sigma^\infty O\langle 4n-1\rangle \arrow[r] \arrow[d, "\varphi"] & \Sigma^{-1}\tau_{\geq 4n}ko \arrow[d] \\ \Sigma^\infty K(\mathbb{Z}, 4n-1) \arrow[r] & \Sigma^{4n-1} H\mathbb{Z}. \end{tikzcd}$$

surj. in range $\rightsquigarrow$

surj:

Serre.

SES:

$$0 \longrightarrow H^*(\Sigma D_2(\Sigma^{-1}\tau_{\geq 4n}ko)) \longrightarrow H^*(\Sigma^{-1}\tau_{\geq 4n}ko)$$

$$\downarrow$$

$$H^*(O\langle 4n-1\rangle)$$

$$\downarrow$$

$$0$$

Ext_A has $\mathbb{Z}/2$

in $(0, 8n-1)$

goes to $(1, 8n-1)$

for $* \leq 8n-1$. In $E_2(O\langle 4n-1\rangle)$

contribute to

$t-s \leq 8n-3$.

nothing at $t-s = 8n-2$.

$\square$