Buildings

Talk 1. Basic Notion

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" Buildings are simplicial complexes satisfying certain axioms, we also call these "abstract buildings".

We begin with some examples

Expl 1 (Flag complex of vector spaces)

k field, V n-dim k-vec. sp

$P_0(V) := \{ W \text{ sub vec.sp of } V \mid W \neq 0, W \neq V \}$

Define a simplicial complex $\Delta(V)$:

vertices : $W \in P_0(V)$

1-simplices : $W_1 \subsetneq W_2$

$\vdots$

m-simplices : $W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{m-1}$

$\vdots$

We say $\Delta(V)$ the building associated to V

The maximal simplices in $\Delta(V)$ are called chambers

$W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{n-1}$, $\dim W_i = i$

$\rightsquigarrow$ $(n-1)$ vertices, simplex of dim $n-2$

" Buildings are simplicial complexes contructed via a given group G, a simplex C + some data, c.f. Tits ICM 1974 "

Contr: Let C be a simplex, G be group. To each simplex σ of C, we assign a subgroup G_σ of G s.t. $G_\tau = \cap G_{\partial\tau}$

$\exists$ a unique minimal G-simplicial complex $\bar{C}$ together with an inclusion $C \to \bar{C}$ s.t.

- G action on $\bar{C}$ is simplicial and each face of C represents exactly one orbit.
- For $\sigma \in C$, G_σ is the stabiliser subgroup of $i(\sigma)$ in $\bar{C}$

$\boxed{G \times C/\sim \text{ where } (g,x) \sim (g',x') \text{ iff } x = x',\ g^{-1}g' \in G_x}$

$G \times C/\sim$ is a simplicial complex with

$\left| \bigsqcup_{x \in C_0} G/G_x \times \{x\} \right|$ vertices, $\left| \bigsqcup_{e \in C_1} G/G_e \times \{e\} \right|$ 1-simplices

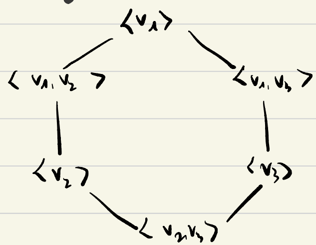
$\cdots$ $\left| \bigsqcup_{\sigma \in C_n} G/G_\sigma \times \{\sigma\} \right|$ n-simplices $\cdots$

$G \curvearrowright (G \times C/\sim)$ by permuting the left cosets.

We say $\Delta(G)$ is the building associated to $(G, C, \mathcal{F})$

Consider the case $n=3$, then $\dim(\Delta(V)) = 1$

Note that $\Delta(V)$ contains lots of subcomplexes that hexagons.



for every unordered basis $\{v_1, v_2, v_3\}$ of V

We call these hexagons apartments in $\Delta(V)$

The first concrete example: $k = \mathbb{F}_2$, $V = \mathbb{F}_2^3$ $(n=3)$

$\rightsquigarrow$ $\Delta(V)$ is a 1-dim simplicial complex

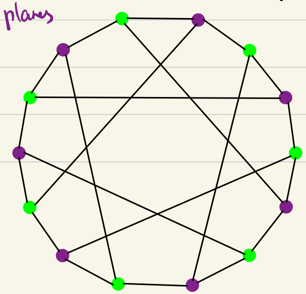
$\mathbb{F}_2^3$ has 7 lines and 7 planes

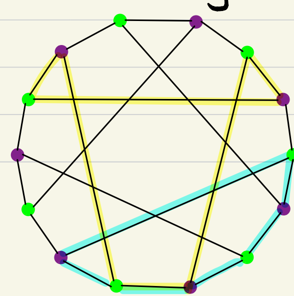
each line lies in exactly 3-planes

each plane contains exactly 3-lines

● lines

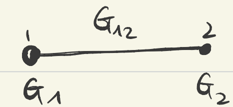
● planes





28 hex.

Expl 1' let $G = SL_3(\mathbb{F}_2)$ $C =$



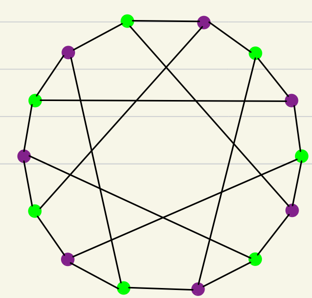
where $G_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$ $G_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$

So, $G_{12} = G_1 \cap G_2$ upper triangular matrices

- $|G_i| = 24$ $|SL_3(\mathbb{F}_2)/G_i| = 168/24 = 7$, $i = 1, 2$

  $|G_{12}| = 8$ $\rightsquigarrow$ $|SL_3(\mathbb{F}_2)/G_{12}| = 21$

- Looking at the preimages of each element of the maps $SL_3(\mathbb{F}_2)/G_{12} \to SL_3(\mathbb{F}_2)/G_i$, $i = 2, 1$,
  - Each edge has one vertex with stabiliser subgr. conjugate of G_1 and one vertex with stabiliser subgr. conjugate of G_2
  - Each vertex has deg 3

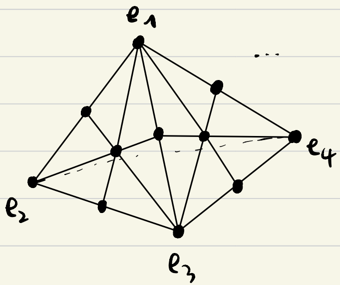


● vertex with Stab G_1

● vertex with Stab G_2

Back to general case n, k.

- Every basis $\{e_1, \dots, e_n\}$ determines a subcomplex, called apartments, via the poset of non-empty vec. spaces of dim $\leq n-1$, spanned by subsets of $\{e_1, \dots, e_n\}$
- Each apartment is isomorphic to the barycentric subdivision of the boundary of an $(n-1)$-simplex

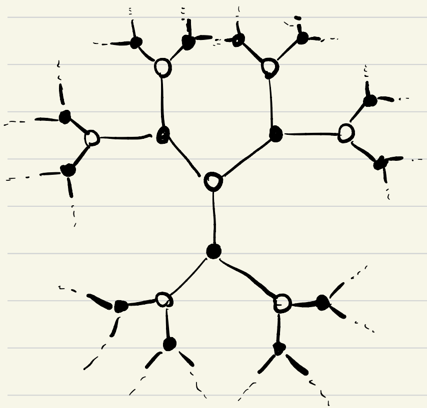


- any two chambers are contained in a common appartment.

Rmk. In Expl 1, $k = \mathbb{F}_2$, $V = \mathbb{F}_2^3$, The simplicial complex $\Delta(V)$ admits a simplicial $SL_3(\mathbb{F}_2)$ action s.t.

- $SL_3(\mathbb{F}_2)$ sends lines to lines, planes to planes, edges to edges
- The stabilizer subgr. for a line is a conjugate of G_1
- The stabiliser subgr. for a plane is a conjugate of G_2.
- —————— edge —————— of G_{12}

Expl 2. Δ be the infinite tree where each vertices has deg 3.



two types of vertices ○ ●

chambers ○—●

Apartment: infinite triangulated lines (in both directions)

Why we care about such trees ?!

$\langle s,t \mid s^2, t^2 \rangle$ ○—●

Expl 2' k field with discrete valuation with residue field $\mathbb{F}_2$. $\mathcal{O}_k$ the ring of integers of k. π a uniformizing element.

Take $G = SL_2(k)$ C: ●————————●

$SL_2(\mathcal{O}_k)$ $\quad$ $G \cap \begin{pmatrix} \mathcal{O}_k & \pi^{-1}\mathcal{O}_k \\ \pi\mathcal{O}_k & \mathcal{O}_k \end{pmatrix}$

Obtain a complex as example 2

Expl 3. (Coxeter group and complex)

Def: A coxeter group is a group G which admits the following representation

$$\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

with $m_{ij} \in \mathbb{N} \cup \{\infty\}$ $m_{ii} = 2$, $m_{ij} \geq 2$.

$m_{ij} = \infty$ means no relation of the form m_{ij} is imposed.

- We say $(G, \{r_1, \dots, r_n\})$ a coxeter system.
- We call $(m_{ij})_{i \geq 1, j \geq 1}$ the Coxeter matrices in case no $m_{ij} = \infty$.

Expl 3 (Coxeter group and coxeter complex)

Def. • A coxeter group is a group G admits the follwing presentation

$$\langle r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

with $m_{ij} \in \mathbb{N} \cup \{\infty\}$, $m_{ii} = 1$, $m_{ij} \geq 2$ for $i \neq j$

If $m_{ij} = \infty$ means that no relation of the form $(r_i r_j)^*$ is imposed.

• We say $(G, \{r_1, \dots, r_n\})$ a Coxeter System

• If $m_{ij} \neq \infty$, for all i, j, then $(m_{ij})_{i,j \geq 1}$ the coxeter matrix. A symmetric matrix

Contr. (Coxeter complex)

$G = \langle r_1, \dots, r_n \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle$

C a $(n-1)$-simplex with vertices labelled $1, \dots, n$

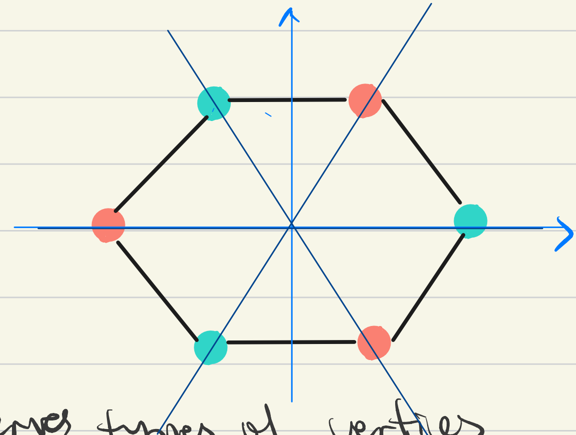
Assign to each vertex $i \in C_0$, the subgroup $G_i := \langle r_j \mid j \neq i \rangle \leq G$

$\rightsquigarrow$ $\Delta(G) := G \times C/\sim$ is the coxeter complex associated to G

<u>Example</u> Consider Dihedral $D_3 = \langle s, t \mid s^2, t^2, (st)^3 \rangle$

$C = \overset{s}{\underset{1}{\bullet}}$ —— $\overset{t}{\underset{2}{\bullet}}$

$\rightsquigarrow$ $G_s = \langle t \mid t^2 = 1 \rangle$ $\quad G_t = \langle s \mid s^2 = 1 \rangle$ $\quad G_{st} = \{e\}$



- D_3 preserves types of vertices
- D_3 acts transitively on the edges
- $\Delta(D_3)$ is a simplicial 1-Sphere.

D_3 acts as a group of symmetries

§ Definition and Properties of abstract buildings

Sit. Let Δ be a finite dimensional simplicial complex where all maximal simplices have the same dimension.

Def. i) The maximal simplices in Δ are called chambers

ii) Two chamber σ, τ are adjacent if they have a common codimension 1 face.

iii) A gallery from a chamber σ to a chamber τ is sequence of chambers

$$(\delta = \delta_0, \delta_1, \dots, \delta_\ell = \tau)$$ Gallery of length ℓ

s.t. δ_{i-1} and δ_i adjacent, $i = 1 \dots \ell$)

vi) We say Δ is a chamber complex if any two chambers can be connected by a gallary.

v) A chamber complex is thick if every simplex of codimension 1 is a face of at least chambers.

vi) A chamber complex is thin ____________________ exactly 2 chambers.

Def We say Δ is a <u>building</u> if it can be expressed as the union of a family of subcomplexes Σ, called apartments, satisfying.

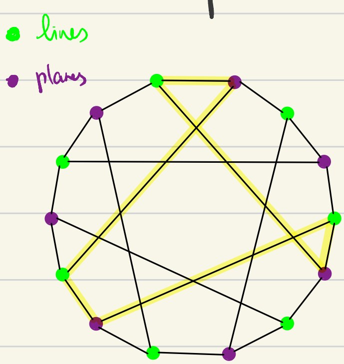
(B0) Each apartment is a Coxeter complex of the same dimension as Δ

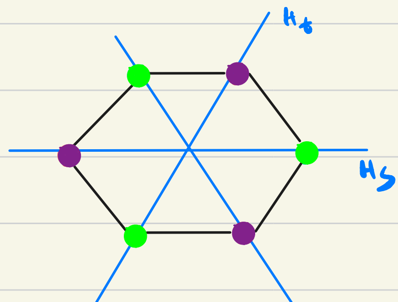
(B1) Any two simplices of Δ are contained in a common apartment

(B2) Given two apartments Σ, Σ' with a common chamber, there is an isomorph. $\Sigma \xrightarrow{\cong} \Sigma'$ fixing $\Sigma \cap \Sigma'$ vertexwise.

Def Let Δ be a building. A <u>system of apartments</u> of Δ is a family of apartments satisfying (B0–B2)

<u>Example</u>. In example 1. The apartments are hexagons.





<u>Consequences</u> Let Δ be a building with a system $\mathcal{A}$ of apartments.

<u>Prop 1</u> All apartments in $\mathcal{A}$ are isomorphic.

<u>Pf</u>. Σ_1, Σ_2 apartments. By (B1) $\exists\, \Sigma$ containing

a chamber of Σ_1 and a chamber of Σ_2

$\rightsquigarrow \quad \Sigma_1 \cong \Sigma \cong \Sigma_2$

Prop. 2 Let $\mathcal{B}$ be another systems of apartments of Δ. Then $\Sigma_{\mathcal{A}} \cong \Sigma_{\mathcal{B}}$ for every $\Sigma_{\mathcal{A}} \in \mathcal{A}$ and $\Sigma_{\mathcal{B}} \in \mathcal{B}$

Idea. Coxeter matrices associated to Δ, defined in Lec. 2 captures exactly the isomorphism type of apartments in Δ

Prop. 3 If the apartments are finite complexes then Δ admits a unique system of apartments.

Prop 3' In general, there is a unique maximal system of apartments.