MORAVA K-THEORY HOMOLOGY OF $\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}, m)$

YUQING SHI

This is the my talk notes at the workshop MIT Talbot 202One on "Ambidexterity in Chromatic Homotopy Theory". In this talk, I'll show Ravenel–Wilson's calculation of $\mathrm{K(n)}_*\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}, m)$, where $\mathrm{K(n)}$ denotes the n-th Morava K-theory spectrum at an odd prime p and $\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}, m)$ denotes the Eilenberg–MacLane space with non-trivial homotopy group $\mathbb{Z}/p^j\mathbb{Z}$ at degree $m \geq 0$. I'll present the computation using the original[1] approach as in [RW80]. Let us first recall the rich structure of $\mathrm{K(n)}_*\left(\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}, m)\right)$.

Situation. We fix an odd prime p throughout the talk. Denote by $\mathrm{K(n)}$ the Morava K-theory with $\mathrm{K(n)}_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ and $\deg(v_n) = 2p^n - 2$.

1. Algebra structures of $\mathrm{K(n)}_*(\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}), m)$

First, let us recall the structure of $\mathrm{K(n)}_*\mathrm{K}_m$ with $\mathrm{K}_m := \mathrm{K}(\mathbb{Z}/p^j\mathbb{Z}, m)$.

Proposition 1.1.

i) *For fixed $j \geq 0$ and $m \geq 0$, the diagonal map of K_m induces a (cocommutative) $\mathrm{K(n)}_*$-coalgebra structure on $\mathrm{K(n)}_*\mathrm{K}_m$. We denote its comultiplication by*
$$\psi_m \colon \mathrm{K(n)}_*\mathrm{K}_m \to \mathrm{K(n)}_*\mathrm{K}_m \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*\mathrm{K}_m.$$

ii) *For fixed $j \geq 0$ and $m \geq 0$, $\mathrm{K(n)}_*\mathrm{K}_m$ is an abelian group object in $\mathbf{coAlg}_{\mathrm{K(n)}_*}$, where the "group addition" is induced by the H-space structure of K_m, and is denoted by*
$$*_m \colon \mathrm{K(n)}_*\mathrm{K}_m \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*\mathrm{K}_m \to \mathrm{K(n)}_*\mathrm{K}_m.$$
In other words, $\mathrm{K(n)}_*\mathrm{K}_m$ is a bicommutative $\mathrm{K(n)}_*$-Hopf algebra

iii) *For fixed $j \geq 0$, the cup product pairing $\mathrm{K}_i \times \mathrm{K}_m \to \mathrm{K}_{i+m}$ induces an "multiplication"*
$$\circ_{i,m} \colon \mathrm{K(n)}_*(\mathrm{K}_i) \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*\mathrm{K}_m \to \mathrm{K(n)}_*(\mathrm{K}_{i+m}),$$
for $i, m \geq 0$. This multiplication is (graded) commutative, unital and distribute over $*$. □

Notation 1.2. Denote by $\mathbf{HopfAlg}_{\mathrm{K(n)}_*}$ the category of (bicommutative) $\mathrm{K(n)}_*$-Hopf algebras.

Proposition 1.3. *For fixed $j \geq 0$, the collection $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is a graded commutative monoid in $\mathbf{HopfAlg}_{\mathrm{K(n)}_*}$, also known as a $\mathrm{K(n)}_*$-*Hopf ring. □

Recall from the previous lecture that $\mathbf{HopfAlg}_{\mathrm{K(n)}_*}$ is equipped with a symmetric monoidal structure, with tensor product denoted by $\boxtimes$. Furthermore, we can consider the subcategory $\mathbf{HopfAlg}_{\mathrm{K(n)}_*, p^j}$ of $\mathrm{K(n)}_*$-Hopf algebras annihilated by multiplication by p^j, for every $j \geq 0$. The subcategory $\mathbf{HopfAlg}_{\mathrm{K(n)}_*, p^j}$ inherits the symmetric monoidal product $\boxtimes$ and has symmetric monoidal unit $\mathrm{K(n)}_*[\mathbb{Z}/p^j\mathbb{Z}] = \mathrm{K(n)}_*\mathrm{K}_0$.

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y.shi@uu.nl.

[1] An alternative proof is presented in [HL, Section 2]

Corollary 1.4. *For fixed $j \geq 0$, the object $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is contained in $\mathbf{HopfAlg}_{\mathrm{K(n)}_*,p^j}$.* □

The main goal of the talk is to give a sketch of the following theorem.

Theorem 1.5 (Ravenel–Wilson)**.** *For fixed $j \geq 0$, the Hopf ring $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is the free $\mathrm{K(n)}_*\mathrm{K}_0$-Hopf ring on the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$, i.e. we have*

$$\oplus_{m\geq 0}\,\mathrm{K(n)}_*\mathrm{K}_m = \mathrm{K(n)}_*\mathrm{K}_0 \oplus \mathrm{K(n)}_*\mathrm{K}_1 \oplus \left(\mathrm{K(n)}_*\mathrm{K}_1 \boxtimes_{\Sigma_2} \mathrm{K(n)}_*\mathrm{K}_1\right) \oplus \cdots . \tag{1.1}$$

Remark 1.6. In the situation of the above theorem, we have $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$ for algebra generators $a_{(i)}, a_{(j)} \in \mathrm{K(n)}_*\mathrm{K}_1$, see [RW80, Lemma 9.1, Lemma 11.2]. Thus, $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ becomes the exterior $\mathrm{K(n)}_*\mathrm{K}_0$-Hopf ring generated by $\mathrm{K(n)}_*\mathrm{K}_1$.

2. INTERPRETATION IN TERMS OF DIEUDONNÉ MODULES

Before proving Theorem 3.1, let me first explain how to translate it in the language of Dieudonné modules, as stated in the previous lecture. For this purpose, we need to work with perfect fields.

Definition 2.1. Define the *cyclic graded Morava* K*-theory* $\overline{\mathrm{K(n)}}_{\bar{t}}(-) := \mathrm{K(n)}_t(-)$ where $\bar{t} \in \mathbb{Z}/(2p^n-2)\mathbb{Z}$ is the reduction of $t \in \mathbb{Z}$. Note that we have $\overline{\mathrm{K(n)}}_* \cong \mathbb{F}_p$.

Recall from Lecture 10 that the Dieudonné ring $D_{\mathbb{F}_p}$ is isomorphic to $\mathbb{Z}_p[F,V]/(FV=p)$ where F denotes the Frobenius and V denotes the Verschiebung. A Dieudonné module over $\mathbb{F}_p$ is a module over the ring $D_{\mathbb{F}_p}$. Recall also the symmetric monoidal functor DM_+ (Lecture 11) which assigns a $\mathbb{F}_p$-Hopf algebra a Dieudonné module.

Notation 2.2. Denote the Hopf algebra $\overline{\mathrm{K(n)}}_*(\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z},1))$ by H_j and the associated Dieudonné module by D_j.

Construction 2.3. We can apply DM_+ to both sides of formula (1.1) and obtain

$$\mathrm{DM}_+\left(\oplus_{m\geq 0}\overline{\mathrm{K(n)}}_*\mathrm{K}_m\right) = \mathbb{Z}/p^j\mathbb{Z} \oplus D_j \oplus D_j \boxtimes_{\Sigma_j} D_j \oplus \cdots .$$

Denote the right hand side by $\wedge_\boxtimes D_j$, the *free Dieudonné algebra* generated by D_j. In particular, we have $\mathrm{DM}_+\left(\overline{\mathrm{K(n)}}_*\mathrm{K}_m\right) \cong \wedge^m_\boxtimes D_j = \left(D_j^{\boxtimes m}\right)_{\Sigma_m}$, for $m \geq 0$.

Therefore, it suffices to study the Dieudonné module structure on D_j and on $\wedge_\boxtimes D_j$, in order to understand the Dieudonneé module $\mathrm{DM}_+\left(\mathrm{K(n)}_*\mathrm{K}_m\right)$ associated to the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_m$. To understand D_j, we consider the the Hopf algebra isomorphism

$$H^\vee := \varprojlim \overline{\mathrm{K(n)}}^*\left(\mathrm{K}(\mathbb{Z}/p^j\mathbb{Z},1)\right) \cong \overline{\mathrm{K(n)}}^*\left(\mathrm{K}(\mathbb{Q}_p/\mathbb{Z}_p,1)\right) \cong \overline{\mathrm{K(n)}}^*\mathrm{K}(\mathbb{Z},2),$$

where

- i) the second isomorphism is induced by $\mathbb{Q}_p/\mathbb{Z}_p \cong \varinjlim_{j\geq 0} \mathbb{Z}/p^j\mathbb{Z}$, and
- ii) the third isomorphism follows from the fact that $\mathrm{K}(\mathbb{Q}_p/\mathbb{Z}_p,1)$ and $\mathrm{K}(\mathbb{Z},2)$ are $\mathrm{K(n)}$-local equivalent[2].

Since $\mathrm{K(n)}$ is complex oriented, we know that $H^\vee \cong \overline{\mathrm{K(n)}}_*[[t]] \cong \mathbb{F}_p[[t]]$. Note that the notation $H^\vee$ means the dual of the $\overline{\mathrm{K(n)}}_*$-Hopf algebra $H := \overline{\mathrm{K(n)}}_*\left(\mathrm{K}(\mathbb{Z},2)\right)$.

Proposition 2.4. *As a $\mathbb{Z}_p$-module, we have $\mathrm{DM}_+(H) \cong \mathbb{Z}_p[V,F]/(FV=p, V^{n-1}=F)$.*

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[2] One sees this by considering the long exact sequence of $\mathrm{K(n)}$-cohomology associated fibre sequence $\mathrm{K}(\mathbb{Q},1) \to \mathrm{K}(\mathbb{Q}/\mathbb{Z},1) \to \mathrm{K}(\mathbb{Z},2)$.

Corollary 2.5. *For fixed $j \geq 0$, the Dieudonné module $D_j = \mathrm{DM}_+(H_j)$ is isomorphic to $\mathbb{Z}/p^j\mathbb{Z}[V,F]/(FV = p, V^{n-1} = F)$.* ☐

Sketch of Proposition 2.4. Over the field $\mathbb{F}_p$, there is a one-to-one correspondence

$$\begin{aligned}\{\text{Formal groups of finite height}\} &\xleftrightarrow{1:1} \{\text{Dieudonné modules of finite type}\}\\ f &\mapsto M\\ \text{Frobenius} &\mapsto \text{Verschiebung}\\ \text{height} &\mapsto \text{rank}\\ \text{dimension} &\mapsto \text{length of the module } M/VM\end{aligned}$$

One can check that the characteristic polynomial of Frobenius, height and dimension of $\operatorname{Spf} H^\vee$ matches the characteristic polynomial of Verschiebung, rank and length of the quotient module of $\mathbb{Z}_p[V,F]/(FV = p, V^{n-1} = F)$. Furthermore, $\operatorname{Spf} H^\vee$ is uniquely determined uniquely by its height and the characteristic polynomial of the Frobenius, since it is of dimension 1. For more details, see [BL07, Section 9]. ☐

Now it remains to study the Frobenius and Verschiebung action on $\wedge_{\boxtimes} D_j$ (Construction 2.3).

Notation 2.6. As a free $\mathbb{Z}/p^j\mathbb{Z}$-module, D_j is generated by $\alpha_{n-1} := 1$, $\alpha_{n-2} := V$, $\dots$, $\alpha_{n-k-1} := V^k$, $\dots$, $\alpha_0 := V^{n-1}$.

Proposition 2.7. *We have*

- i) $V\alpha_0 = p\alpha_{n-1}$,
- ii) $V\alpha_i = \alpha_{i-1}$, *for* $i \geq 1$, *and*
- iii) $F\alpha_i = V^{n-1}\alpha_i$, *for* $i \geq 0$.

Proof. We use the relations $VF = p$ and $V^{n-1} = F$. ☐

Recall from Remark 1.6 that $\wedge_{\boxtimes} D_j$ is the exterior algebra generated by D_j. In other words, $\wedge_{\boxtimes} D_j = \oplus_{m=0}^n \wedge_{\boxtimes}^m D_j$. Considering the $\mathbb{Z}/p^j\mathbb{Z}$-exterior algebra $\wedge \underline{D_j}$ with $\underline{D_j}$ the free $\mathbb{Z}/p^j\mathbb{Z}$-module underlying the Dieudonné module D_j.

Proposition 2.8. *The exterior algebra $\wedge \underline{D_j}$ admits a $D_{\mathbb{F}_p}$-module structure where V and F acts on $\wedge^m \underline{D_j}$, for every $m \geq 0$, via the formulas*

$$V(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_m}) = V(\alpha_{i_1}) \wedge \cdots \wedge V(\alpha_{i_m}),$$

$$F\big(V(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_s}) \wedge \alpha_{i_{s+1}} \wedge \cdots \wedge \alpha_{i_m}\big) = \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_s} \wedge F(\alpha_{i_{s+1}} \wedge \cdots \wedge \alpha_{i_m}),$$

for every tuple $(i_1, i_2, \dots, i_m) \in \mathbb{N}^m$ with $0 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$.

Sketch. We can construct the Verschiebung and Frobenius actions inductively using Proposition 2.7 and the formulas

$$\begin{aligned}V(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_m}) &= V(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_{m-1}}) \wedge \alpha_{i_m - 1},\\ F(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_m}) &= \alpha_{i_1+1} \wedge F(\alpha_{i_2} \wedge \alpha_{i_3} \wedge \cdots \wedge \alpha_{i_m}).\end{aligned}$$

To check that it is a well-defined Dieudonné module, see [BL07, Section 10]. ☐

It turns out the Dieudonné module structure on $\wedge_{\boxtimes} D_j$ "coincide" with the one on $\wedge \underline{D_j}$.

Theorem 2.9. *For any $1 \leq m \leq n$, there are isomorphisms of Dieudonné modules*

$$\begin{aligned}\wedge^0 \underline{D_j} &\to \wedge_{\boxtimes}^0 D_j, \ 1 \to 1,\\ \wedge^m \underline{D_j} &\to \wedge_{\boxtimes}^m D_j, \ \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_m} \mapsto \alpha_{i_0} \circ \cdots \circ \alpha_{i_m}.\end{aligned}$$

3. Proof of the Theorem 3.1

Let me first recall the statement of the theorem, note that $\mathrm{K}_m = \mathrm{K}\left(\mathbb{Z}/p^j\mathbb{Z}, m\right)$.

Theorem 3.1 (Ravenel–Wilson)**.** *For fixed $j \geq 0$, the Hopf ring $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is the free $\mathrm{K(n)}_*\mathrm{K}_0$-Hopf ring on the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$.*

So, to prove the theorem, we need to show that

i) the Hopf ring $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is generated by $\mathrm{K(n)}_*\mathrm{K}_1$

ii) The relations in the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_m$, for every $m \geq 0$, are a consequence of axioms of the Hopf ring and the Hopf algebra structure of $\mathrm{K(n)}_*\mathrm{K}_1$.

We will demonstrate the ideas of the proof of the theorem in the case where $j = 1$. The proofs for the $j \geq 2$ cases are exactly the same, see [RW80, Section 11, 12].

Situation 3.2. In the rest of the text, we set $\mathrm{K}_m := \mathrm{K}\left(\mathbb{Z}/p\mathbb{Z}, m\right)$. Recall that p is a fixed odd prime.

3.1. The Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$.

As a first step, we would like to study the $\mathrm{K(n)}_*$-Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$. Note that the Eilenberg–MacLane space K_1 fits into a fibre sequence

$$\mathrm{K}_1 \xrightarrow{\delta} \mathrm{K}(\mathbb{Z},2) \xrightarrow{\times p} \mathrm{K}(\mathbb{Z},2). \tag{3.1}$$

We will use the Hopf algebra structure of $\mathrm{K(n)}_*\mathrm{K}(\mathbb{Z},2)$ to obtain the one on $\mathrm{K(n)}_*\mathrm{K}_1$. Recall that $\mathbb{C}\mathrm{P}^\infty \simeq \mathrm{K}(\mathbb{Z},2)$.

Proposition 3.3.

i) *As $\mathrm{K(n)}_*$-algebras, we have $\mathrm{K(n)}^*\mathbb{C}\mathrm{P}^\infty \cong \mathrm{K(n)}_*[[c]]$ with $\deg c = 2$.*

ii) *As $\mathrm{K(n)}_*$-modules, we have $\mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty \cong \mathrm{K(n)}_*[\beta_0, \beta_1, \ldots]$ with $\deg \beta_i = 2i$. The module generators β_i, for $i \geq 0$, are determined by the $\mathrm{K(n)}_*$-cohomology-homology pairing $\langle c^i, \beta_j\rangle = \delta_{ij}$, for every $i, j \geq 0$.*

iii) *Set $\beta_{(i)} := \beta_{p^i}$ and $\beta_{(i)} := 0$ for $i < 0$. There is an isomorphism $\mathrm{K(n)}_*$-algebras*

$$\mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty \cong \mathrm{K(n)}_*[\beta_{(0)}, \beta_{(1)}, \ldots, \beta_{(k)}, \ldots]/\, \beta^{*p}_{(n+i-1)} = v_n^{p^i}\beta_{(i)},$$

where $*$ denotes the algebra operation in $\mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty$ generated by the H-space structure of $\mathbb{C}\mathrm{P}^\infty$.

iv) *The comultiplication ψ on $\mathrm{K(n)}_*(\mathbb{C}\mathrm{P}^\infty)$ is given by*

$$\psi(\beta_k) = \sum_{i=0}^{m} \beta_i \otimes \beta_{k-i}.$$

The following theorem determines the Hopf algebra structure of $\mathrm{K(n)}_*\mathrm{K}_1$.

Theorem 3.4.

i) *The induced map $\delta_*\colon \mathrm{K(n)}_*\mathrm{K}_1 \to \mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty$ is a Hopf algebra monomorphism.*

ii) *As a $\mathrm{K(n)}_*$-module, we have $\mathrm{K(n)}_*\mathrm{K}_1 \cong \mathrm{K(n)}_*[a_0, a_1, \ldots, a_{p^n-1}]$ with $\deg a_k = 2k$ and $\delta_*(a_k) = \beta_k$ for $0 \leq k < p^n$.*

Notation 3.5. Denote $a_{(i)} := a_{p^i}$ and $a_{(i)} := 0$ for $i < 0$.

Recall that the commutative algebra multiplication of the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_m$ is denote by $*$ (Proposition 1.1), for every $m \geq 0$.

Corollary 3.6. *There is $\mathrm{K(n)}_*$-algebra isomorphism*

$$\mathrm{K(n)}_*\mathrm{K}_1 \cong \mathrm{K(n)}_*[a_{(0)}, a_{(1)}, \ldots, a_{(n-1)}]/\, a^{*p}_{(n+i-1)} = v_n^{p^i} a_{(i)}.$$

The comultiplication ψ *on* $\mathrm{K(n)}_*(\mathbb{C}\mathrm{P}^\infty)$ *is given by*

$$\psi(a_k) = \sum_{i=0}^{k} a_i \otimes a_{k-i},$$

for $0 \leq k \leq p^{n-1}$. □

Sketch of the proof of Theorem 3.4. Consider the Gysin sequence

$$\cdots \to \mathrm{K(n)}_*\mathrm{K}_1 \xrightarrow{\delta_*} \mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cap e_\delta} \mathrm{K(n)}_{*-2}\mathbb{C}\mathrm{P}^\infty \to \cdots \tag{3.2}$$

$$y \mapsto y \cap [p]_{\mathrm{K(n)}}(c) \tag{3.3}$$

$$\beta_{n+i} \mapsto \beta_i, \tag{3.4}$$

associated to the fibre sequence $S^1 \to \mathrm{K}_1 \xrightarrow{\delta} \mathrm{K}(\mathbb{Z}, 2)$ induced by the fibre sequence 3.1. Here, e_δ denotes the Euler class of the "sphere bundle" δ and $[p]_{\mathrm{K(n)}}$ is the p-series of $\mathrm{K(n)}$. We have $e_\delta = [p]_{\mathrm{K(n)}}(c)$ because of the following homotopy pullback diagram

$$\begin{tikzcd}
\mathrm{K}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S(\gamma^1) \arrow[d] \\
\mathbb{C}\mathrm{P}^\infty \arrow[r, "\times p"] & \mathbb{C}\mathrm{P}^\infty,
\end{tikzcd}$$

where $S(\gamma^1)$ denotes the sphere bundle associated to the tautological line bundle γ^1 of $\mathbb{C}\mathrm{P}^\infty$. By formula 3.4 we see that the map $\cap e_\delta$ is surjective. Thus, the long exact sequence 3.2 splits into short exact sequences

$$0 \to \mathrm{K(n)}_*\mathrm{K}_1 \xrightarrow{\delta_*} \mathrm{K(n)}_*\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cap e_\delta} \mathrm{K(n)}_{*-2}\mathbb{C}\mathrm{P}^\infty \to 0.$$

So, δ_* is monomorphism and we can read off its image using 3.4. □

Remark 3.7. We can rewrite the algebra isomorphism in Corollary 3.6 as

$$\mathrm{K(n)}_*\mathrm{K}_1 \cong \mathrm{K(n)}_*[a_{(1)}, a_{(2)}, \ldots, a_{(n-1)}]/\sim$$

where we quotient out by the equivalence relation given by $a_{(i)}^{*p} = 0$, for $1 \leq i \leq n-2$ and $a_{(n-1)}^{*p^2} = 0$.

3.2. Computation of $\mathrm{K(n)}_*\mathrm{K}_m$ for $n > 1$.

Recall we wish to prove that the Hopf ring $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ is freely generated by the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$. So, I'll introduce a notation for tensor products of elements of $\mathrm{K(n)}_*\mathrm{K}_1$.

Notation 3.8. For $I = (i_1, i_2, \ldots, i_k, \ldots, i_m)$ with $0 \leq i_k \leq n-1$, we define $a_I \in \mathrm{K(n)}_*\mathrm{K}_m$ via the iterated cup product pairing (Proposition 1.1)

$$\circ^m : \mathrm{K(n)}_*\mathrm{K}_1 \boxtimes \cdots \boxtimes \mathrm{K(n)}_*\mathrm{K}_1 \to \mathrm{K(n)}_*\mathrm{K}_m$$

$$a_{i_1} \boxtimes \cdots \boxtimes a_{i_m} \mapsto a_I := a_{i_1} \circ \cdots \circ a_{i_m}$$

We mentioned at the beginning (Remark 1.6) of the talk that the Hopf ring multiplication on $\oplus_{m\geq 0}\mathrm{K(n)}_*\mathrm{K}_m$ encodes an exterior algebra structure. This is because of the following proposition.

Proposition 3.9. *For* $a_{(i)}, a_{(j)} \in \mathrm{K(n)}_*\mathrm{K}_1$ *with* $0 \leq i < n$ *and* $0 \leq j < n$*, we have*

i) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)} = 0$*, and*

ii) $a_{(i)} \circ a_{(i)} = 0$

Sketch. The first statement follows from axioms of Hopf rings and $\chi(a_{(i)}) = -a_{(i)}$ in the Hopf algebra $\mathrm{K(n)}_*\mathrm{K}_1$. The second statement follows from the first one and our convention that p is an odd prime. For more details, see [RW80, Lemma 9.1]. □

To prove the main theorem (Theorem 3.1), it suffices to verify the following theorem. Denote by $I_n \coloneqq (0, 1, \dots, n-1)$ and $\mathbb{I}_m \coloneqq \{(i_1, i_2, \dots, i_m) \mid 0 < i_1 < i_2 < \cdots\cdots i_m < n\}$, for every $m \geq 1$.

Theorem 3.10. *We have* $\mathrm{K(n)}_*$*-algebra isomorphisms*

i) $\mathrm{K(n)}_*\mathrm{K}_0 \cong \mathrm{K(n)}_*[\mathbb{Z}/p\mathbb{Z}]$,
ii) $\mathrm{K(n)}_*\mathrm{K}_l \cong \mathrm{K(n)}_*$, *for* $l > n$,
iii) $\mathrm{K(n)}_*\mathrm{K}_n \cong \mathrm{K(n)}_*[a_{I_n}]/\,(a_{I_n}^{*p} + (-1)^n v_n a_{I_n})$, *and*
iv) $\mathrm{K(n)}_*\mathrm{K}_m \cong \bigotimes_{I\in\mathbb{I}_m} \mathrm{K(n)}_*[a_I]/\,a_I^{*\rho(I)}$ *for* $m < n$, *where the tensor product is over* $\mathrm{K(n)}_*$ *and* $\rho(I) = 1 + \max\left(\{0\} \cup \{s+1 \mid i_{m-s} = n-1-s\}\right)$.

Remark 3.11. As a corollary of the above theorem, the $\mathrm{K(n)}_*$-coalgebra structure of $\mathrm{K(n)}_*\mathrm{K}_m$ is obtained from the coalgebra structure of $\mathrm{K(n)}_*\mathrm{K}_1$ and the the cup product paring map $\circ$ (which is a $\mathrm{K(n)}_*$-coalgebra morphism), for every $m \geq 2$.

In the proof of Theorem 3.10, we need to use the following proposition. Recall that $\overline{\mathrm{K(n)}}$ denotes the cyclic graded Morava K-theory (Definition 2.1), and recall Dieudonné module structure on $\mathrm{DM}_+(\overline{\mathrm{K(n)}}_*\mathrm{K}_m) = \bigwedge^m_{\boxtimes} D_1$ (Proposition 2.8).

Proposition 3.12. *In* $\overline{\mathrm{K(n)}}_*\mathrm{K}_m$, *the* V *and* F *action on* $a_I = a_{(i_1)} \circ \cdots a_{(i_m)}$ *is the same as the one on* $\alpha_I = \alpha_{(i_1)} \circ \cdots \circ \alpha_{(i_m)} \in \bigwedge^m_{\boxtimes} D_1$, *for every* $I \in \mathbb{I}_m$.

The idea of proving Theorem 3.10 is by induction. We have the induction base, since part i) of the theorem is straightforward and we fully understood the Hopf algebra structure on $\mathrm{K(n)}_*\mathrm{K}_1$ in Section 3.1. In the remaining time of the talk, I'll introduce the key ingredient of performing the induction step: the Bar spectral sequence.

One way to obtain K_{m+1} is through the Bar construction of K_m, for every $m \geq 0$. In other words, we can think of K_{m+1} as the geometric realisation of the following simplicial space

$$\mathrm{K}_{m+1} = \mathrm{BK}_m = \varinjlim \left(\cdots \mathrm{K}_m \otimes \mathrm{K}_m \otimes \mathrm{K}_m \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathrm{K}_m \otimes \mathrm{K}_m \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathrm{pt} \right).$$

Hence, we obtain a tower of cofibrations

$$\mathrm{pt} = \mathrm{B}_0\mathrm{K}_m \subseteq \mathrm{B}_1\mathrm{K}_m \subseteq \cdots \subseteq \mathrm{B}_s\mathrm{K}_m \subseteq \mathrm{B}_{s+1}\mathrm{K}_m \subseteq \cdots \subseteq \mathrm{K}_{m+1}$$

where $\mathrm{B}_s\mathrm{K}$ denotes the s-truncated geometric realisation

$$\mathrm{B}_s\mathrm{K}_m = \varinjlim \left(\mathrm{K}_m^{\otimes s} \cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathrm{K}_m \otimes \mathrm{K}_m \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathrm{pt} \right).$$

As a consequence, we can consider the $\mathrm{K(n)}_*$-homology spectral sequence associated to the tower of cofibrations.

Theorem 3.13. *There is a spectral sequence* $(E^r_{*,*}(\mathrm{K}_m), d^r)_{r\geq 1}$ *of* $\mathrm{K(n)}_*$*-Hopf algebras converging to the* $\mathrm{K(n)}_*\mathrm{K}_{m+1}$ *such that*

i) $E^1_{s,t}(\mathrm{K}_m) \cong \widetilde{\mathrm{K(n)}}_{s+t}(\mathrm{B}_s\mathrm{K}_m/\mathrm{B}_{s+1}\mathrm{K}_{s+1})$, *and*
ii) $E^2_{s,t}(\mathrm{K}_m) \cong \mathrm{Tor}_{s,t}^{\mathrm{K(n)}_*\mathrm{K}_m}(\mathrm{K(n)}_*, \mathrm{K(n)}_*)$.

We have several remarks regarding this theorem.

Remark 3.14.

i) Theorem 3.13.i) follows from the construction of the spectral seqeunce.

ii) To see part ii) of the theorem, we rewrite the E_1-page as

$$\begin{aligned} E^1_{s,*} &\cong \widetilde{\mathrm{K(n)}}_* \left(\mathrm{B}_s\mathrm{K}_m/\mathrm{B}_{s+1}\mathrm{K}_{s+1}\right) \\ &\cong \widetilde{\mathrm{K(n)}}_* \left(\Sigma^s\, \mathrm{K}_m^{\wedge s}\right) \\ &\cong \left(\widetilde{\mathrm{K(n)}}_*\mathrm{K}_m\right)^{\otimes_{\mathrm{K(n)}_*} s}, \end{aligned} \tag{3.5}$$

where the second isomorphism follows from the equivalence $\mathrm{B}_s\mathrm{K}_m/\mathrm{B}_{s+1}\mathrm{K}_{s+1} \simeq \Sigma^s\, \mathrm{K}_m^{\wedge s}$ and the last isomorphism is a result of Künneth isomorphism. Thus, we see that $E^1_{s,*}$ is the bar resolution computing $\mathrm{Tor}_{s,t}^{\mathrm{K(n)}_*\mathrm{K}_m}(\mathrm{K(n)}_*, \mathrm{K(n)}_*)$.

Remark 3.15. An important point of Theorem 3.13 is that the spectral sequence is a Hopf algebra spectral sequence, meaning that each page is a (graded) $\mathrm{K(n)}_*$-Hopf algebra and the differential $d^r\colon E^r_{*,*} \to E^r_{*,*}$ is a Hopf algebra derivation, for every $r \geq 1$. This provides us with useful properties of the spectral sequence which simplifies the computations. For example,

i) For every $r \geq 1$, the differential d^r satisfies the Leibnitz rules

$$d^r(x * y) = x * d^r y + d^r x * y,$$

and the "co-Leibnitz" rule (as a coalgebra spectral sequence)

$$(d^r \otimes 1 + 1 \otimes d^r)\psi = \psi d^r.$$

ii) As a corollary of i), an element of the lowest homological degree supporting a non-trivial differential must be an algebra generator.

iii) The target of every differential must be primitive, see for example [Smi70, p.78, Lemma].

Remark 3.16. The cup product pairing induces a pairing of spectral sequences

$$\circ_{i,m}\colon E^r_{s,*}(\mathrm{K}_i) \otimes_{\mathrm{K(n)}_*} E^r_{s,*}(\mathrm{K}_m) \to E^r_{s,*}(\mathrm{K}_{i+m}),$$

for every $i, m \geq 0$. Under this pairing, we have $d^r(x \circ y) = d^r x \circ y$. This would imply that the differentials in the spectral sequence $E^r_{s,*}(\mathrm{K}_m)_{r\geq 0}$ can be computed inductively from the differentials of $E^r_{s,*}(\mathrm{K}_{m-1})_{r\geq 0}$.

3.3. A baby example of the Bar spectral sequence.

Let us take a look at the first non-trivial example of the Bar spectral sequence. Our example takes place in the following situation.

Situation 3.17. Let $n = 2$, $p = 3$ and thus $2p^n - 2 = 16$.

Recall from Remark 3.7 that we have $\mathrm{K(2)}_*$-algebra isomorphism

$$\mathrm{K(2)}_*\mathrm{K}_1 \cong \mathrm{K(2)}_*\left[a_{(1)}\right]/a_{(1)}^{*9}$$

with $\deg a_{(1)} = 6$ (Theorem 3.4). One can compute the E_2-page of the spectral sequence $E^r_{*,*}(\mathrm{K}_1)$ by writing down an explicit free resolution. Abbreviate the Hopf algebra $\mathrm{Tor}_{*,*}^{\mathrm{K(2)}_*\mathrm{K}_1}(\mathrm{K(2)}_*, \mathrm{K(2)}_*)$ by $\mathrm{H}_{*,*}$. We have

$$E^2_{*,*}(\mathrm{K}_1) \cong \Lambda_{\mathrm{K(2)}_*}\left(\sigma a_{(1)}\right) \otimes_{\mathrm{K(2)}_*} \Gamma_{\mathrm{K(2)}_*}\left(\phi\left(a_{(1)}^{*3}\right)\right) \tag{3.6}$$

where

i) $\Lambda_{\mathrm{K(2)}_*}\left(\sigma a_{(1)}\right)$ denotes the exterior algebra generated by $\sigma a_{(1)}$ with $\sigma a_{(1)} \in \mathrm{H}_{1,6}$,

ii) $\Gamma_{\mathrm{K}(2)_*}\left(\phi\left(a^{*3}_{(1)}\right)\right)$ denotes the divided power algebra generated by $\gamma_1 := \phi\left(a^{*3}_{(1)}\right)$ with $\gamma_1 \in H_{2,54}$.
iii) Define $\gamma_i \in \Gamma_{\mathrm{K}(2)_*}(\gamma_1)$, for $i \geq 2$, inductively via $\gamma_i\gamma_j = \binom{i+j}{i}\gamma_{i+j}$. Recall that $\Gamma_{\mathrm{K}(2)_*}(\gamma_1)$ is a free $\mathrm{K}(2)_*$-module generated by γ_i, for $i \geq 1$. Furthermore, the algebra generators of $\Gamma_{\mathrm{K}(2)_*}(\gamma_1)$ are γ_{3^j} for $j \geq 0$.

Remark 3.18. In the formula 3.6,
i) the element $\sigma a_{(1)}$ is called "suspension" of $a_{(1)}$, which is in general in $\mathrm{H}_{1,\deg a_{(1)}}$, and
ii) the element $\phi(a^{*p}_{(1)})$, called "transpotence" of $a^{*p}_{(1)}$, lives in general in $\mathrm{H}_{2,p^2 \deg a_{(1)}}$.
See [RW80, Lemma 6.6] for more details.

By degree reason, Remark 3.15.ii) and iii), we can verify that the first non-trivial differentials appear on the E^5-page and is generated by $d^5\colon E^5_{6,k} \to E^1_{1,k-1}$. Ravenel–Wilson shows further that this differentials d^5 is nontrivial by using the Verschiebung and Frobenius actions (Proposition 3.12). See Figure 3.3 for a illustration of the E_5-page of this spectral sequence[3].

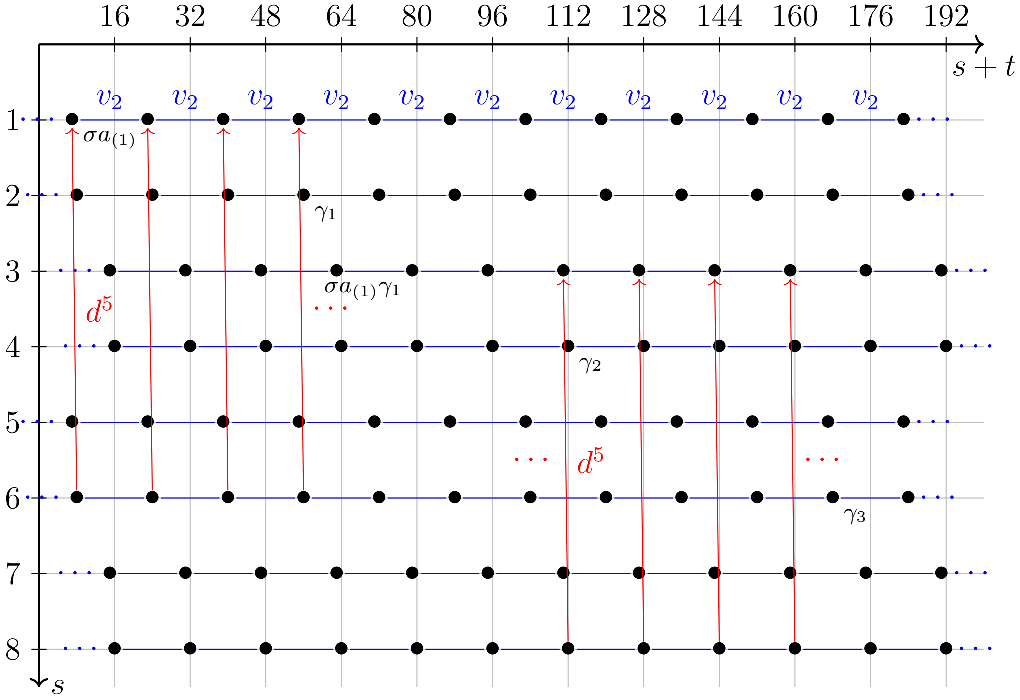

FIGURE 1. An illustration of the E_5-page of the spectral sequence $E^r_{*,*}(\mathrm{K}_1)$. Each black dots represents a copy of $\mathbb{F}_p$. The horizontal blue line connecting adjacent nodes indicates multiplication by v_2. On each s-coordinate, the v_2-multiplication extends infinitely to the left and to the right. The differentials is of degree $(-1,-5)$. Every element with s-coordinate 5 or 7 is also hit by a d^5-differential, which we don't draw in the picture.

Using the Hopf algebra structure of the spectral sequence, one can propogate the differentials $d^5\colon E^5_{k,6} \to E^5_{k-1,6}$. It turns out that there is no room for other differentials after we considered the differentials generated by d^5. In particular, the only elements that are not hit by the d^5 are the elements in the second and the fourth line. As a result, these

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[3] Thanks to Pablo Magni for the tex codes of the spectral sequeunces.

are exactly the elements that survices to E^∞-page. We show the E^∞-page of the spectral sequence in Figure 3.3.

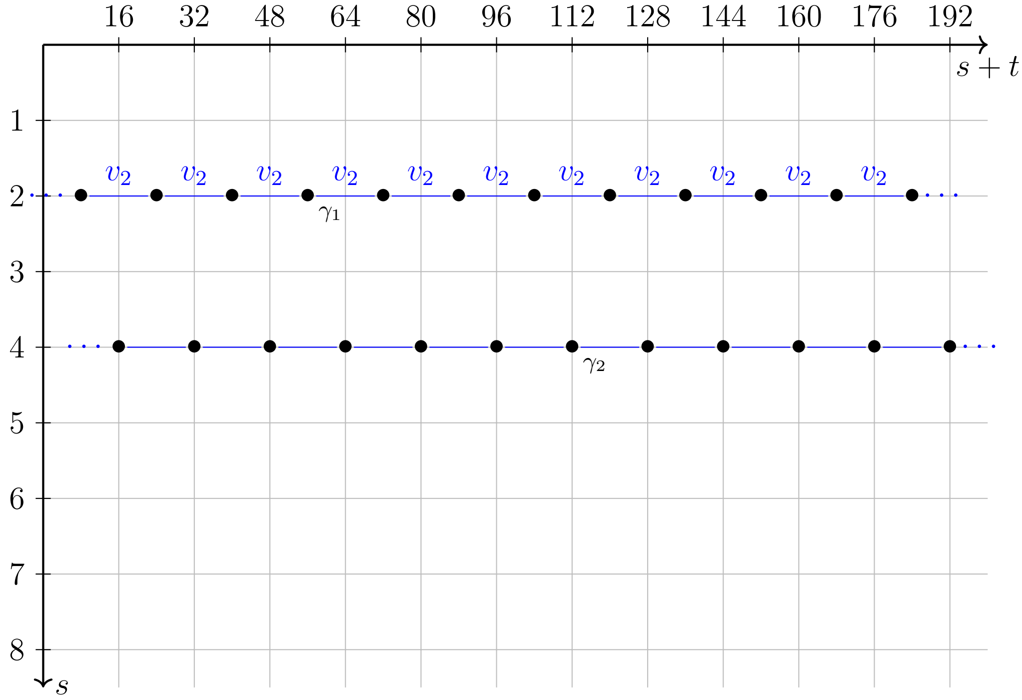

FIGURE 2. The $E\infty$-page of the spectral sequence.

To complete the example, we remark that

i) The element $a_{(0,1)} \in \mathrm{K}(2)_*\mathrm{K}_2$ is represented by $v_2^{-3}\gamma_1$, see [RW80, Lemma 9.7], and

ii) one can show that $a^{*3}_{(0,1)} = v_2 a_{(0,1)}$ using again the Verschiebung and Frobenius action on the Hopf algebra $\overline{\mathrm{K}(2)}_*\mathrm{K}_2$, see [RW80, Theorem 9.2.c)].

Therefore, we have and $\mathrm{K}(2)_*$-algebra isomorphism

$$\mathrm{K}(2)_*\mathrm{K}_2 \cong \mathrm{K}(2)_*[a_{(0,1)}]/a^{*3}_{(0,1)} = v_2 a_{(0,1)}.$$

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