PROOF OF THICK SUBCATEGORY THEOREM

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In this talk, our main focus is to explain the proof of the thick subcategory theorem, *cf.* Theorem 2.7. We will begin with a very brief introduction of the Spanier–Whitehead duality, which is used often in proofs in [HS98]. The main reference for this talk is [HS98] and [Rav92, Chapter 5].

1. Spanier–Whitehead duality

The original idea of Spanier–Whitehead duality is that the stable homotopy type of a compact subset $W \subseteq \mathrm{S}^n$ determines the stable homotopy type of the complement $\mathrm{S}^n \setminus W$. For a detailed geometric intuition, see [Ada74].

Definition 1.1. Let E be a spectra. We define the *Spanier–Whitehead dual* $\mathrm{D}\,E$ of E to be the function spectrum $\mathrm{F}(E, \mathbb{S})$.

Theorem 1.2. *Let F be a finite spectrum. Then the Spainier–Whitehead dual $\mathrm{D}\,F$ of F has the following property.*

i) *We can consider F as the suspension spectrum of a finite CW-complex $\underline{F}$ embedded in S^N for some $N \in \mathbb{N}$. Then $\mathrm{D}\,F$ is also finite, and it is a suitable (depends on N) suspension of the suspension spectrum of the complement $\mathrm{S}^N \setminus \underline{F}$.*
ii) *We have $\mathrm{D}\,\mathrm{D}\,F \simeq F$.*
iii) *For a homology theory h_*, there is a natural isomorphism between $\mathrm{h}_k(F)$ and $\mathrm{h}^{-k}(\mathrm{D}\,F)$ for $k \in \mathbb{Z}$.*
iv) *We have $\mathrm{D}(F \wedge E) = \mathrm{D}\,F \wedge \mathrm{D}\,E$ for every spectrum E.*

Let $f\colon E \to E' \in \mathbf{Sp}$. The (adjoint to) the induced map on Spanier–Whitehead dual can be written in the form $\mathrm{D}\,f\colon \mathbb{S} \to E' \wedge \mathrm{D}\,E$. In particular, for the map $\mathrm{id}_E\colon E \to E$, its Spanier–Whitehead dual is

$$\mathrm{D}\,\mathrm{id}_E\colon \mathbb{S} \to E \wedge \mathrm{D}\,E.$$

We are going to use the following proposition in the proof of Theorem 2.7.

Proposition 1.3. *Let $X \in \mathbf{Sp}^{\mathrm{fin}}$ and $n \in \mathbb{N} \cup \{\infty\}$. The map $\mathrm{D}\,\mathrm{id}_X\colon \mathbb{S} \to X \wedge \mathrm{D}\,X$ induces a nonzero homomorphism in $\mathrm{K(n)}$-homology if and only if $\mathrm{K(n)}_* X \neq 0$.*

Sketch. The map $\mathrm{K(n)}_*(\mathrm{D}\,\mathrm{id}_X)$ of finite dimensional $\mathrm{K(n)}_*$-vector spaces is of the form

$$\begin{aligned} \mathrm{K(n)}_* &\to \mathrm{K(n)}_*(X) \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*(\mathrm{D}\,X) \\ 1 &\mapsto \sum_{i \in I} e_i \otimes e_i^\vee, \end{aligned}$$

where we identity $\mathrm{K(n)}_*(\mathrm{D}\,X)$ with $(\mathrm{K(n)}_*(X))^\vee$ and $\{e_i\}_{i \in I}$ denotes a basis for the vector space $\mathrm{K(n)}_*(X)$. □

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2. Proof of the thick subcategory theory

Notation 2.1. Let $m \in \mathbb{N}$. Denote by $(-)^{(m)}$ the m-fold smash products.

Let us recall the definition of a thick subcategory.

Definition 2.2. A full subcategory $\mathcal{C}$ of $\mathbf{Sp}^{\mathrm{fin}}$ is *thick* if it is closed under weak equivalences, cofiber sequences and retracts, *i.e.*

i) If $X \in \mathcal{C}$ and $Y \simeq X$, then $Y \in \mathcal{C}$;
ii) If $X \to Y \to Z$ is a cofibre sequence in $\mathbf{Sp}^{\mathrm{fin}}$ and two of $\{X, Y, Z\}$ are in $\mathcal{C}$, then so is the third;
iii) If $X \in \mathcal{C}$ and Y is a retract of X, then $Y \in \mathcal{C}$.

Remark 2.3. Let $X, Y \in \mathbf{Sp}^{\mathrm{fin}}$ and $\mathcal{C}$ be a thick subcategory of $\mathbf{Sp}^{\mathrm{fin}}$. It follows from Definition 2.2.iii) that if $X \vee Y \in \mathcal{C}$, then $X \in \mathcal{C}$ and $Y \in \mathcal{C}$.

Remark 2.4. We will see in Magdalena's talk that we can define a thick subcategory of a general triangulated subcategory in a similar manner.

Example 2.5. Let $n \in \mathbb{N} \cup \{\infty\}$. Denote by $\mathbf{Sp}^{\mathrm{fin}}_{(p),n}$ the full subcategory of $\mathbf{Sp}^{\mathrm{fin}}$ whose objects are $\mathrm{K(n-1)}$-acyclic spectra. It is not hard to check that $\mathbf{Sp}^{\mathrm{fin}}_{(p),n}$ is a thick subcategort of $\mathbf{Sp}^{\mathrm{fin}}$. We omit the proof here.

Proposition 2.6. *Let $\mathcal{C}$ be a thick subcategory of $\mathbf{Sp}^{\mathrm{fin}}$. If $X \in \mathcal{C}$ and $Y \in \mathbf{Sp}^{\mathrm{fin}}$, then $X \wedge Y \in \mathcal{C}$.*

Proof. Any finite spectrum is a *finite* colimit of suspensions of the sphere spectrum. It suffices then to prove that thick subcategories are closed under finite colimits and smashing with suspensions of the sphere supectrum. □

Theorem 2.7 (Thick subcategory theorem)**.** *If $\mathcal{C} \subseteq \mathbf{Sp}^{\mathrm{fin}}_{(p)}$ is a thick subcategory, then $\mathcal{C} = \mathbf{Sp}^{\mathrm{fin}}_{(p),n}$ for some $0 \leq n \leq \infty$.*

For the proof of this theorem, we need to use a refined version of the nilpotence theorem, and a useful cofibre sequence, *cf.* Corollary 2.12. Let me recall these first.

Definition 2.8. A map $f\colon E \to E' \in \mathbf{Sp}$ is *smash nilpotent* if $f^{(n)}\colon E^{(m)} \to (E')^{(m)}$ is null-homotopic for large enough m.

Theorem 2.9 (Nilpotence theorem)**.** *A map $f\colon F \to E \in \mathbf{Sp}$ with $F \in \mathbf{Sp}^{\mathrm{fin}}$ and $E \in \mathbf{Sp}_{(p)}$ is smash nilpotent if and only if $\mathrm{K(n)}_*(f) = 0$ for all $0 \leq n \leq \infty$.*

Proof. We will prove this in the future talks. You can also find a proof in [Rav92, Theorem 5.1.4]. □

Corollary 2.10. *Let $F, Z \in \mathbf{Sp}^{\mathrm{fin}}$ and $E \in \mathbf{Sp}$ and R is a ring spectrum. We have*

i) *If a map $f\colon F \to E \wedge R$ satisfies $\mathrm{K(n)}_* f = 0$ for all $0 \leq n \leq \infty$, then the composite*

$$\phi\colon F^{(m)} \xrightarrow{f^{(m)}} (E \wedge R)^{(m)} \cong E^{(m)} \wedge R^{(m)} \xrightarrow{\mathrm{id}_{E^{(m)}} \wedge \mu_R} E^{(m)} \wedge R$$

is null-homotopic for large enough m. Here μ_R denotes the multiplication of R.

ii) *A map $g\colon F \to E$ has the property that*

$$g^{(m)} \wedge \mathrm{id}_Z\colon F^{(m)} \wedge Z \to E^{(m)} \wedge Z$$

is null-homotopic for large enough m if and only if

$$\mathrm{K(n)}_*(g \wedge \mathrm{id}_Z) = 0$$

for all $0 \leq n \leq \infty$.

Proof. Part i) follows from Theorem 2.9: f is smash nilpotent.

The proof for par ii) is the following: If $g^{(m)} \wedge \mathrm{id}_Z$ is null-homotopic for large enough m, the induced map

$$\mathrm{K(n)}_*(g^{(m)} \wedge \mathrm{id}_Z)\colon\ (\mathrm{K(n)}_*(F))^{(m)} \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*(Z) \to (\mathrm{K(n)}_*(E))^{(m)} \otimes_{\mathrm{K(n)}_*} \mathrm{K(n)}_*(Z)$$

is the null map of $\mathrm{K(n)}_*$-vector spaces, for all $0 \leq n \leq \infty$. Here $(\mathrm{K(n)}_*(F))^{(m)}$ and $(\mathrm{K(n)}_*(E))^{(m)}$ denotes m-fold tensor product over $\mathrm{K(n)}_*$. Thus we have that $\mathrm{K(n)}_*(g) = 0$ for $0 \leq n \leq \infty$. Therefore $\mathrm{K(n)}_*(g \wedge \mathrm{id}_Z) = 0$ for all $0 \leq n \leq n$.

For the "if"-direction, the Spainer–Whitehead dual of the map $g \wedge \mathrm{id}_Z$ is

$$g'\colon F \to E \wedge Z \wedge \mathrm{D}\, Z$$

via Spanier–Whitehead duality

Thus we have that $\mathrm{K(n)}_*(g') = 0$ for all $0 \leq n \leq \infty$.

Note that $Z \wedge \mathrm{D}\, Z$ is a ring spectrum. Thus applying part i), we have that the composite

$$F^{(m)} \to E^{(m)} \wedge Z \wedge \mathrm{D}\, Z,$$

which is the Spainer–Whitehead dual of the map $g^{(m)} \wedge \mathrm{id}_Z$, is null-homotopic for large enough m. Therefore, $g^{(m)} \wedge \mathrm{id}_Z$ is null-homotopic for large enough m.

□

The following are two pieces of elementary stable homotopy theory.

Lemma 2.11. *Let $E, E', E'' \in \mathbf{Sp}$, and $E \xrightarrow{f} E' \xrightarrow{g} E''$ be a sequence of maps. The map $\mathrm{cofib}(f) \to \mathrm{cofib}(g \circ f)$ induced by g gives rise to a cofibre sequence*

$$\mathrm{cofib}(f) \to \mathrm{cofib}(g \circ f) \to \mathrm{cofib}(g).$$

Proof. See [Ada74, Lemma III.6.8]. □

Corollary 2.12. *Let $f \in E \to E'$ and $g \in E'' \to E'''$ be maps of spectra. There is a cofibre sequence*

$$E \wedge \mathrm{cofib}(g) \to \mathrm{cofib}(f \wedge g) \to \mathrm{cofib}(g) \wedge E'.$$

Proof. Identify $f \wedge g$ with $(f \wedge \mathrm{id}_{E'}) \circ (\mathrm{id}_E \wedge g)$ and apply Lemma 2.11. □

Proof of Theorem 2.7. To proof this theorem, it suffices to show that

Claim 2.13. *Let $X \in \mathcal{C}$ and $Y \in \mathbf{Sp}^{\mathrm{fin}}$. If $\mathrm{supp}(Y) \subseteq \mathrm{supp}(X)$, then $Y \in \mathcal{C}$.*

Let $Z \in \mathbf{Sp}^{\mathrm{fin}}$. Recall from Tommy's talk, we have

$$\mathrm{supp}(Z) = \{n \in \mathbb{N} \cup \{\infty\} \mid \mathrm{K(n)}_*(Z) \neq 0\}.$$

It follows from the claim that

$$\mathcal{C} = \mathbf{Sp}^{\mathrm{fin}}_{(p),n} \ \text{ with } n := \min\{m \in \mathbb{N} \cup \{\infty\} \mid \exists X \in \mathcal{C},\ \mathrm{K(m)}_*(X) \neq 0\}.$$

Proof of the Claim 2.13. Consider the fibre sequence

$$F \xrightarrow{f} \mathbb{S} \xrightarrow{\mathrm{D}\,\mathrm{id}_X} X \wedge \mathrm{D}\, X$$

Note that $\mathrm{cofib}(f) \simeq X \wedge \mathrm{D}\, X$. By Proposition 2.6, we have that $Y \wedge \mathrm{cofib}(f) \in \mathcal{C}$. Let us set $g = f^{(m-1)}$ and apply Corrolary 2.12, we obtain a cofiber sequence

$$Y \wedge F \wedge \mathrm{cofib}\left(f^{(m-1)}\right) \to Y \wedge \mathrm{cofib}\left(f^{(m)}\right) \to Y \wedge \mathrm{cofib}(f) \wedge F^{(m-1)}.$$

Thus by induction, we have that

$$Y \wedge \mathrm{cofib}\left(f^{(m)}\right) \in \mathcal{C}$$

for all $m \in \mathbb{N}$.

By Proposition 1.3, we have that $\mathrm{K(m)}_* f \neq 0$ if and only if $m \notin \operatorname{supp}(X)$. Therefore,

$$\mathrm{K(m)}_* \left(\mathrm{id}_Y \wedge f\right) = 0, \ \forall m \in \mathbb{N}.$$

Indeed, if $m \notin \operatorname{supp}(Y)$, we have the $\mathrm{K(m)}_*(Y) = 0$; if $m \in \operatorname{supp}(Y) \subseteq \operatorname{supp}(X)$, we have $\mathrm{K(m)}_*(f) = 0$.

By Corollary 2.10, we have that $\mathrm{id}_Y \wedge f^m$ is null-homotopic for m large enough. Thus we have

$$Y \wedge \operatorname{cofib}\left(f^{(m)}\right) \simeq \operatorname{cofib}\left(\mathrm{id}_Y \wedge f^{(m)}\right) \simeq Y \vee \left(\Sigma\left(Y \wedge F^{(m)}\right)\right).$$

Thus Y is a retract of $Y \wedge \operatorname{cofib}\left(f^{(m)}\right) \in \mathcal{C}$. Therefore $Y \in \mathcal{C}$. □

□

Remark 2.14. Actually, the thick subcategory theorem is equivalent to the nilpotence theorem. There is a proof in [HS98, Section 4.5].

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill.-London, 1974. Chicago Lectures in Mathematics.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. Math. (2)*, 148(1):1–49, 1998.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.